

ME 201/MTH 281/ME 400/CHE 400

ASSIGNMENT #6 2009

Assignments handed in by 6 PM on Thursday Oct. 22 will receive a 5 point bonus. Assignments handed in after that but by 4 PM on Friday Oct. 23 will receive full credit but no bonus. No assignments will be accepted after 4 PM on Oct. 23.

LECTURE SCHEDULE AND READING

<u>Section in Class Notes</u>	<u>Date</u>	<u>Section in Text</u>
3. Separation of Variables, Part I		
3.6 Beyond Fourier Series	Th Oct 15	---
4. Sturm Liouville Theory		
4.1 Eigenfunctions and Eigenvalues	F, M Oct 16, 19	5.1-5.3, 5.5
4.2 Eigenfunction Expansions	W Oct 21	5.3

PROBLEM

(1) (100 points) The object of this assignment is to make use of Sturm-Liouville theory to carry out a complete analysis of a separation of variables solution of a modified version of the heat equation. The problem, as described below, is broken up into a number of steps. At each step, you are given the result that you are asked to show, so that you can proceed through the problem even if you get stuck on a particular step.

Statement of the Problem

This problem requires the solution of the heat equation for a slab, in a case where the thermal properties depend on position. The equation in that case has the general form

$$\rho C \frac{\partial T}{\partial t} = \frac{\partial}{\partial x} \left(K \frac{\partial T}{\partial x} \right), \quad (1)$$

where the density ρ , the specific heat per unit mass C , and the thermal conductivity K may all depend on x . In this particular problem, the x -interval is $0 \leq x \leq L$, and the thermal properties are given by

$$\rho = \rho_0, \quad C = C_0 \left(1 + \frac{x}{L} \right), \quad K = \frac{K_0}{1 + x/L}, \quad (2)$$

where ρ_0, C_0, K_0 are all constant. The boundary conditions and initial conditions are

$$T(0,t) = 0, \quad T(L,t) = 0, \quad T(x,0) = T_0. \quad (3)$$

where T_0 is constant.

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(a) Scaling (10 points)

Introduce the dimensionless quantities (verify that they are dimensionless)

$$\hat{x} = 1 + x/L, \hat{T} = T/T_0, \hat{t} = D_0 t/L^2, \text{ where } D_0 = K_0/\rho_0 C_0, \quad (4)$$

and show that the problem becomes

$$\hat{x} \frac{\partial \hat{T}}{\partial \hat{t}} = \frac{\partial}{\partial \hat{x}} \left(\frac{1}{\hat{x}} \frac{\partial \hat{T}}{\partial \hat{x}} \right), 1 < \hat{x} < 2, \hat{T}(1, \hat{t}) = \hat{T}(2, \hat{t}) = 0, \hat{T}(\hat{x}, 0) = 1. \quad (5)$$

(b) Separation of Variables (30 points)

From here on, we use only the dimensionless formulation (5). To make the writing easier, we drop the hats over the letters. Try a separated solution of the form $T = G(t)F(x)$, and show that F and G satisfy

$$\frac{dG}{dt} = -\lambda G, \quad \frac{d}{dx} \left(\frac{1}{x} \frac{dF}{dx} \right) = -\lambda x F, \quad 1 < x < 2. \quad (6)$$

The boundary conditions on F are $F(1) = 0, F(2) = 0$. From the equation for $F(x)$, prove that the eigenvalues are all positive.

Verify that the general solution of the second of equations (6) is given by

$$F(x) = C_1 \cos\left(\frac{\sqrt{\lambda} x^2}{2}\right) + C_2 \sin\left(\frac{\sqrt{\lambda} x^2}{2}\right). \quad (7)$$

(c) Eigenvalues and Eigenfunctions (30 points)

Show that the eigenvalues and eigenfunctions are given by

$$\lambda_n = \frac{4n^2 \pi^2}{9}, \quad F_n = \sin\left(\frac{n\pi(x^2 - 1)}{3}\right), \quad n = 1, 2, 3, \dots \quad (8)$$

The general theory says that these eigenfunctions must be orthogonal with respect to an appropriate weight function. Verify the truth of that.

(d) Solution of Initial Value Problem (30 points)

Solve the dimensionless initial value problem given by (5) in terms of the eigenfunctions F_n . Use Mathematica to generate plots of your dimensionless solution as a function of x for various times. Find a one-term long-time approximation. Find the dimensional e-folding time for the first mode.

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CHALLENGE PROBLEM

In this problem, you are asked to carry out further calculations with the solution you found above. We restore the hats to the dimensionless variables here.

A general expression for the (dimensional) thermal energy in the slab (per unit face area of the slab) is

$$E(t) = \int_0^L \rho C T(x,t) dx.$$

Here x and t are our original dimensional variables. Use the dimensionless variables to show that this is equal to $\rho_0 C_0 T_0 L \int_1^2 \hat{x} \hat{T}(\hat{x}, \hat{t}) d\hat{x}$. Use this expression and your solution from problem 1 above to find a series for the quantity $\frac{E(t)}{E(0)}$. Plot this ratio as a function of the dimensionless time $\hat{t} = D_0 t / L^2$.

The dimensional heat flux leaving through the face at $x = 0$ is

$$F_{left}(t) = K \left. \frac{\partial T}{\partial x} \right|_0 = \frac{K_0 T_0}{L} \left. \frac{\partial \hat{T}}{\partial \hat{x}} \right|_{\hat{x}=1},$$

and the dimensional flux leaving through the face at $x = L$ is

$$F_{right}(t) = -K \left. \frac{\partial T}{\partial x} \right|_L = -\frac{K_0 T_0}{2L} \left. \frac{\partial \hat{T}}{\partial \hat{x}} \right|_{\hat{x}=2}.$$

From your solution, derive series expressions for F_{left} and F_{right} and verify that

$\frac{dE}{dt} = -F_{left} - F_{right}$, which is the basic global energy balance for the slab. Verify also the result that $F_{left} = F_{right}$. This last result is perhaps somewhat unexpected, given that the problem is not symmetric about the midpoint of the slab.