ME 201/MTH 281/ME 400/CHE 400
ASSIGNMENT #5 2009

Assignments handed in by 6 PM on Thursday Oct. 15 will receive a 5-point bonus. Assignments handed in after that but by 4 PM on Friday Oct. 16 will receive full credit but no bonus. No assignments will be accepted after 4 PM on Oct. 16.

LECTURE SCHEDULE AND READING

Section in Class Notes	Date	Section in Text
3. Separation of Variables, Part I		
3.2 Laplace Equation	W,Th,F Sept 30, Oct 1,2	2.5.1
Fall Break – No Class	M Oct 5	
Exam Review	W Oct 7	---
EXAM #1	Th Oct 8	
3.3 Wave Equation	F Oct 9	4.1-4.4
3.4 Guitar String	M Oct 12	---
3.5 Energy Integrals & Uniqueness	W Oct 14	---

PROBLEMS
3.2 LAPLACE AND LAPLACE-LIKE EQUATIONS

(1) (25 points) Consider the boundary value problem given below for the steady-state temperature *T* in a rectangle. The quantity T_0 appearing in a boundary condition is a constant.

$$\frac{\partial^2 T}{\partial x^2}+\frac{\partial^2 T}{\partial y^2}=0,\ 0<x<a,\ 0<y<b,$$

$$\text{with } \frac{\partial T}{\partial y}(x,0)=0,\ T(x,b)=T_0\frac{x}{a}(1-\frac{x}{a}),\ T(0,y)=0\ \text{, and } T(a,y)=0\ .$$

(a) (10 points) Solve this problem by separation of variables and superposition. For parts (b) and (c) below, use Mathematica and use the values $a=2$ m, $b=1$ m, and $T_0=40$ °C.

(b) (5 points) Check your calculations by comparing selected numerical values of the boundary temperature on $y=b$ with values calculated from your series. As a further check on your work, show from your solution that the temperature at the midpoint of the rectangle is 5.413 °C.

(c) (10 points) Plot lines of constant temperature for 1, 2, 3, 4, 5, 6, 7, 8, and 9 °C on a single contour plot. Use your series to get the values of *T* for the plot.

(2) (25 points) Use separation of variables to solve the boundary value problem given below for a Laplace-like equation in a rectangle.

$$\frac{\partial^2\Phi}{\partial x^2}+\frac{\partial^2\Phi}{\partial y^2}+2\frac{\partial\Phi}{\partial x}=\Phi,\ 0<x<1 \text{ and } 0<y<2,$$

$$\text{with } \Phi(x,0)=0,\ \Phi(x,2)=0,\ \Phi(0,y)=0,\text{ and } \Phi(1,y)=5.$$

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3.3, 3.4 WAVE EQUATION

(3) (25 points) Consider the initial value problem, given below, for the wave equation with zero initial displacement and a given initial velocity.

$$\frac{\partial^2 y}{\partial t^2} = c^2 \frac{\partial^2 y}{\partial x^2},\ 0 < x < L,\ t > 0,\ \text{with } y(0,t) = 0,\ y(L,t) = 0,$$

$$y(x,0) = 0,\ \text{and } \frac{\partial y(x,0)}{\partial t} = V_0 \frac{x}{L}\left(1 - \frac{x}{L}\right).$$

Here V_0 is a positive constant.

(a) (20 points) Use the separated solutions found in class to solve this initial value problem.

(b) (5 points) Use any method you like to estimate the maximum displacement of the string.

3.5 ENERGY INTEGRALS AND UNIQUENESS

(4) (25 points) In class it was shown that the total energy of a stretched string in motion is

$$E = \tfrac{1}{2}\int_0^L \left\{ \sigma \left(\frac{\partial y}{\partial t}\right)^2 + T\left(\frac{\partial y}{\partial x}\right)^2 \right\} dx,$$

where L is the length of the string, T is the tension in the string, and σ is the mass per unit length of the string. We showed that if y is a solution of

$$\frac{\partial^2 y}{\partial t^2} = c^2 \frac{\partial^2 y}{\partial x^2},\ \text{with } y(0,t) = 0 \text{ and } y(L,t) = 0,$$

where $c^2 = \dfrac{T}{\sigma}$, then $\dfrac{dE}{dt} = 0$ – that is the energy is conserved. This is consistent with the undamped motion described by the equation, and with the zero-displacement boundary conditions (there are forces on the string at the endpoints, but if there is no motion there, no work is done by the forces).

In the present problem, you will look at the effect of adding an elastic restoring force to every point of the string. (You can think of the string as having springs attached all along its length.) If the effective spring constant is k, then there is an additional vertical force per unit length at each point of the string, of magnitude $-ky$. The vertical force balance then yields the equation

$$\sigma \frac{\partial^2 y}{\partial t^2} = T\frac{\partial^2 y}{\partial x^2} - ky. \tag{1}$$

We assume that the string is fixed at both ends, so the boundary conditions are still

$$y(0,t) = 0 \text{ and } y(L,t) = 0. \tag{2}$$

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Because the potential energy of a spring is $\frac{1}{2}ky^2$, we may speculate that the total energy is now

$$E = \frac{1}{2}\int_0^L \left\{ \sigma\left(\frac{\partial y}{\partial t}\right)^2 + T\left(\frac{\partial y}{\partial x}\right)^2 + ky^2 \right\} dx. \tag{3}$$

(a) (10 points) By using the equation and the boundary conditions, show that $\frac{dE}{dt} = 0$.

(b) (15 points) Consider the initial value problem defined by (1) and (2) above, and by

$$y(x,0) = f(x), \text{ and } \frac{\partial y(x,0)}{\partial t} = g(x)\ , \tag{4}$$

where f and g are given. Use your result of part (a) to prove that the solution is unique.

CHALLENGE PROBLEM

In class it was shown that the total energy of a stretched string in motion is

$$E[y] = \frac{1}{2}\int_0^L \left\{ \sigma\left(\frac{\partial y}{\partial t}\right)^2 + T\left(\frac{\partial y}{\partial x}\right)^2 \right\} dx, \tag{5}$$

where L is the length of the string, T is the tension in the string, and σ is the mass per unit length of the string. The displacement y is a solution of

$$\frac{\partial^2 y}{\partial t^2} = c^2 \frac{\partial^2 y}{\partial x^2}\text{, with } y(0,t) = 0 \text{ and } y(L,t) = 0, \tag{6}$$

where $c^2 = \frac{T}{\sigma}$. Also in class, we used separation of variables to find the solution of (6) as

$$y(x,t) = \sum_{n=1}^{\infty} y_n(x,t)\text{, where } y_n(x,t) = \{A_n \cos(\omega_n t) + B_n \sin(\omega_n t)\}\sin\left(\frac{n\pi x}{L}\right), \tag{7}$$

with $\omega_n = n\pi c / L$, and with the coefficients A_n and B_n being determined by the initial conditions. We call $y_n(x,t)$ the nth mode of the solution.

(a) (40 points) We define the energy of the nth mode to be $E_n = E[y_n]$. By substituting the expression for y_n into the expression defining E, show that $E_n = \frac{T}{4L}(n\pi)^2(A_n^2 + B_n^2)$. Note that E_n does not depend on time.

(b) (60 points) In this calculation you will find Parseval's theorem (or some equivalent use of orthogonality) to be helpful. Substitute the solution (7) for $y(x,t)$ into the expression (5) for the energy E, and show that $E = \sum_{n=1}^{\infty} E_n$. On the basis of this result, draw a conclusion about how the distribution of energy among the modes changes as the motion proceeds.