

ME 201/MTH 281/ME 400/CHE 400

ASSIGNMENT #2 2009

Assignments handed in by 6 PM on Wednesday September 16 will receive a 5 point bonus. Assignments handed in after that but by 6 PM on Thursday September 17 will receive full credit but no bonus. No assignments will be accepted after 6 PM on Thursday.

In problem (2) and in the challenge problem, you will need to calculate Fourier coefficients. In doing that, you can save some labor by using Mathematica. The last page of this assignment gives an example of such a calculation.

LECTURE SCHEDULE AND READING

<u>Section in Class Notes</u>	<u>Date</u>	<u>Section in Text</u>
1.3 Separation of Variables – A First Attempt	F,W Sept. 4,9	2.1, 2.2, 2.3.1-2.3.5
2.1 Basic Fourier Series	Th,F,M Sept. 10,11,14	3.1

PROBLEMS

1.3 SEPARATION OF VARIABLES

(1) (20 points) Find an explicit solution to the boundary value problem given below.

$$\frac{\partial T}{\partial t} = D \frac{\partial^2 T}{\partial x^2}, \quad T(0,t) = 0, \quad T(L,t) = 0, \quad \text{and}$$

$$T(x,0) = T_0 \sin\left(\frac{\pi x}{L}\right) \cos\left(\frac{\pi x}{L}\right) + T_1 \sin\left(\frac{5\pi x}{L}\right) \cos\left(\frac{3\pi x}{L}\right),$$

where T_0 and T_1 are constants. (Hint: You will need to use some standard trig identities to transform the initial condition.)

(2) (30 points) For the boundary value problem given below for heat conduction in a bar, carry out a separation of variables solution, and take the calculation as far as you can with what we have learned so far. In particular, you should find all of the separated solutions that satisfy the boundary conditions and then attempt to superpose them to satisfy the initial conditions. Your calculation here will resemble the one presented in class in section 1.3 of the class notes. You may find some help from the challenge problem on Assignment #1.

$$\frac{\partial T}{\partial t} = D \frac{\partial^2 T}{\partial x^2}, \quad 0 \leq x \leq L, \quad t \geq 0, \quad \frac{\partial T}{\partial x}(0,t) = 0, \quad T(L,t) = 0, \quad T(x,0) = T_0 \left(\frac{x}{L}\right).$$

This corresponds to the left end of the bar being insulated, the right end being maintained at zero temperature, and an initial temperature which varies linearly from 0 at $x = 0$ to T_0 at $x = L$.

2.1 BASIC FOURIER SERIES

(3) (25 points) Consider the function $f(x)$ defined by

$$f(x) = x|x| \text{ for } -1 < x \leq 1, \text{ and } f(x+2) = f(x) \text{ for all } x.$$

(a) (5 points) Give a sketch of $f(x)$ versus x for $-3 \leq x \leq 3$.

(b) (5 points) Show that f is piecewise smooth.

(c) (10 points) Find the Fourier series for f .

(CONTINUED NEXT PAGE)

(3) (continued)

(d) (5 points) Using the basic convergence theorem, tell what the series converges to for each x in the interval $[-1,1]$.

COMPUTER WORK

For this assignment, you are asked to work through the following five sections of the Mathematica Tutorial: Arithmetic; Algebra; Defining Constants, Expressions and Functions; Plotting; Summing Series. You may download the tutorial from the course web site. After you have loaded the tutorial notebook into Mathematica, you can open any of the sections by double clicking on the square bracket opposite the section name. You do not need to hand in any of your work with the tutorial.

(4) (25 points) In this problem, you will continue working with the series of problem 3.

(a) (5 points) Use Mathematica to graph the function f over the interval $[-3,3]$. (Hint: If you have defined a function $f_{bas}[x]$ over the interval $[-L, L]$, the extension of f_{bas} which is periodic with period $2L$ can be defined in Mathematica by $f[x_] := f_{bas}[\text{Mod}[x, 2*L, -L]]$.)

(b) (10 points) Use the Sum function, which you worked with in the tutorial, to define a function which plots the M^{th} partial sum of the series. By M^{th} partial sum, the following is meant:

$$a_0 + \sum_{n=1}^M \{a_n \cos(n\pi x) + b_n \sin(n\pi x)\}.$$

Your function should take M as an argument, and produce a graph of the above expression over the x -range $[-3,3]$.

(c) (10 points) Use your function of part (b) to produce three plots of partial sums for x in the interval $[-3,3]$: $M = 5$, $M = 10$, and $M = 20$. Show the original function f on each of your three graphs. Do you think 20 terms is enough for the series to represent the original function? (You may either answer that question or criticize it.)

CHALLENGE PROBLEM

In this problem, you will make an experimental (i.e. numerical) study of the convergence of the Fourier series for each of three different functions. All of the functions are periodic with period 2. For each function you are asked to plot (1) the function on the interval $[-3,3]$, and (2) the M^{th} partial sum, as defined in problem (4), for $M = 3, 5$, and 10. The partial sum graphs should also show the original function. The three specific functions are given below. On the basis of your results, try to make some qualitative speculations about the relation between the rate of convergence of the series, and the behavior of the function. In class later this week we will make much more precise statements about that topic.

(a) $f(x) = x$ for $-1 < x \leq 1$, and $f(x+2) = f(x)$ for all x .

(b) $f(x) = x^2$ for $-1 < x \leq 1$, and $f(x+2) = f(x)$ for all x .

(c) $f(x) = x(1-x^2)$ for $-1 < x \leq 1$, and $f(x+2) = f(x)$ for all x .

(CONTINUED NEXT PAGE)

An Example of the Calculation of Fourier Coefficients

We show in this notebook how to use *Mathematica* to calculate the Fourier coefficients of a given function. We take the base function to be

$$f[x_] := x + x^2$$

We take the base interval to be $-L$ to L , with

$$L = 2;$$

We let $a[n]$ be the n th cosine coefficient, and $b[n]$ be the n th sine coefficient. Then

$$a[0] = \frac{1}{2L} \int_{-L}^L f[x] dx$$

$$\frac{4}{3}$$

For $n > 0$, we calculate the Fourier cosine coefficient which we temporarily call `coeff`:

$$\text{coeff} = \frac{1}{L} \int_{-L}^L f[x] \text{Cos}[(n \pi x) / L] dx$$
$$\frac{8 (2 n \pi \text{Cos}[n \pi] + (-2 + n^2 \pi^2) \text{Sin}[n \pi])}{n^3 \pi^3}$$

A little more complicated than we might have expected. The problem is that *Mathematica* does not yet know that n is an integer. We can use a `Simplify` command to tell *Mathematica* that.

```
Simplify[coeff, n ∈ Integers]
```

$$\frac{16 (-1)^n}{n^2 \pi^2}$$

The first argument of `Simplify` is the expression to be simplified, and the second argument tells *Mathematica* to make the assumption that n is contained in the set of integers. The symbol which looks like an epsilon is the set inclusion symbol, and, in *Mathematica*, it is not an epsilon. You can find it on several of the palettes (for example, on the Basic Math Assistant, under Typesetting, in the category labeled x , approximately in the middle of the 5th row).

We can streamline this process, by now defining the n th Fourier cosine coefficient all in one step.

$$a[n_] = \text{Simplify}\left[\frac{1}{L} \int_{-L}^L f[x] \text{Cos}[(n \pi x) / L] dx, n \in \text{Integers}\right]$$

$$\frac{16 (-1)^n}{n^2 \pi^2}$$

Does this definition for general n override our earlier definition of a[0]? We check it:

$$a[0]$$

$$\frac{4}{3}$$

Mathematica gives precedence to function values defined for specific arguments, so our earlier definition of a[0] takes precedence over the generic a[n] when n is zero. (In fact the generic result for a[n] is not even valid when n is zero.)

In the same way, we can calculate the Fourier sine coefficients.

$$b[n_] = \text{Simplify}\left[\frac{1}{L} \int_{-L}^L f[x] \text{Sin}[(n \pi x) / L] dx, n \in \text{Integers}\right]$$

$$-\frac{4 (-1)^n}{n \pi}$$