HOMEWORK

IN SPACE

ME 201/MTH 281
PROJECT 2008
INTRODUCTION

In this project, you are member of a special group of young Starfleet officers, charged with a continuing exploration of the abandoned colony on the planet Glia-6. The specialty of your group is the solution of various boundary value problems that have arisen in the study of the technology of the long-ago inhabitants of the colony. You are presently in orbit about Glia-6, but the planned trip to the surface has been postponed because of a persistent and violent dust storm. To your great surprise, Captain Kirk and Science Officer Spock have given your group a homework problem. After all the excellent work your group has done, you are all more than a little annoyed at being treated as mere students. But orders are orders and the problem must be solved. What you don’t know at this point is that your life may depend on the accuracy of your calculations. The details of all this are given below.

PRACTICAL MATTERS

You may work alone, or in groups of two. You may discuss the project freely with others. If you get significant help from someone else, that fact should appear as an acknowledgment or reference in your report. Any books or articles that you use should be referenced. Your report should be brief but self-contained, so that a reader not having access to these directions will know what you have done.

A typical report organization might be something like the following: Introduction (with a general description of the problem); Formulation (the detailed quantitative formulation of the problem); Results (results of the calculations, including any graphs or tables necessary to make the results clear); Summary (a summary of your conclusions); References (books, articles or people consulted). Mathematica notebooks showing calculations should be included as an Appendix to the report. Although typed reports are preferred, neatly handwritten reports are acceptable. Your project grade will be based on both the write-up (20%) and the technical content (80%). For groups of two, only one report need be submitted.

Most of the information you need to do this project is contained in this writeup. The one exception is the detailed development of the technique to solve the Poisson equation in spherical coordinates. That will be presented in class on Wednesday November 26, the last class before Thanksgiving break. Because some of you may be traveling then, the lecture notes for that class will be put up on the web.

CONNECTION WITH THE REAL WORLD

In the most general terms, the thermal analysis here is typical for a steady-state reactor – the fission reactions generate heat, some of which is removed by the coolant to be used in power generation, and some of which is dumped to the external environment (by forced convection in this case). However, the details of this project are not meant to give a realistic picture of a reactor thermal analysis.
GENERAL BACKGROUND FOR THE PROJECT

The Glia System

Visits of the USS Enterprise to the Glia starsystem began on Stardate 5526.3, with an examination of an ecosystem on Glia-2 (ME163 F93). On that visit Starfleet Academy cadets saved an ecosystem. They were commended even though their bold plan violated the Prime Directive. The second visit to the Glia system, on Stardate 6872.1, involved the launch of an unmanned probe to the large airless planet Glia-4 (ME 163 F94). Showing great ingenuity, the cadets on that mission successfully designed a nonlinear landing suspension for the probe. The discovery on Glia-4 of naturally occurring dilithium crystals (used in warp propulsion) led to the third visit on stardate 7304.6, during which Academy cadets solved a crucial forced vibrations problem, allowing the dilithium crystals to be transported undamaged to the Enterprise (ME163 S97). Because of the importance of the dilithium mining, a colony in a biodome was set up on Glia-4 to continue the mining operation. An epidemic of Jasmine fever on Glia-4 was the occasion of the fourth visit. Once again Academy cadets proved the value of mathematics in modeling events, and developed a vaccine transport strategy which minimized the number of lives lost (ME 163 S98). The fifth adventure in the Glia system (stardate 8378.2, ME163 S99) began innocently as a recreational visit to the abandoned colonies on Glia-6. The situation was instantly transformed into a dire emergency when Captain Kirk was bitten by a feral glion carrying a deadly virus. Only by the most strenuous efforts were the Academy cadets able to save Kirk, by developing an optimal treatment protocol for the viral infection. The sixth training mission to the Glia starsystem (Stardate 8993.1, ME 163 S2000) required cadets to calculate re-entry orbits for a 20th century Apollo capsule, and then, in a surprise test of their computational acumen, to actually ride the capsule down to the surface of Glia-3, accompanied by Professor Clark’s cat Billie. Additional details on some of these earlier visits to the Glia system are available on the web at

http://www.me.rochester.edu/~clark/ME163Web/webproj/proj.html

The seventh visit to the Glia system (Stardate 9604.2, ME201/MTH281 F2000) was the first involving upper level cadets from Starfleet Academy. On that visit, two cadets well-versed in partial differential equations were able to avert a war with the Klingons by their skillful analysis of a cooking problem. The eighth expedition to the Glia system (Stardate 9863.4, ME201/MTH281 F2001), also with upper level cadets, included a visit to the abandoned colonies on Glia-6. There the cadets unraveled the secrets of an acoustic lock and gained access to the historical records of the abandoned colony. During the ninth visit to the Glia system (Stardate 9927.3, ME201/MTH281 F2002), a surface visit to Glia-6 was not possible because of a continuing dust storm. The Academy cadets on board the orbiting Enterprise engaged in the design of a stealth submarine, using the novel technique of flow into and out of the submarine to mask the more prominent components of the far field. On the tenth visit to the Glia starsystem (Stardate 9966.2, ME201/MTH281 F2003), the Cadets were charged with explaining why there was a melt down of a nuclear reactor long ago in the abandoned Platonian colony on Glia-6. Significant new information on the reactor was available in a recently discovered manual in the historical repository opened several years ago by Cadets. The manual describes a unique modification to the boundary condition on the reactor which allows critical states at a lower fuel
density. The Cadets had to reproduce the critical value analyses given in the manual, and then search for an explanation of the meltdown. On the eleventh visit to the Glia system, a group of young Starfleet officers are left on Glia-6 to explore the old Platonian science museum. The exploration unexpectedly turns into a life-threatening entombment, which can end only if the group can solve the difficult thermal problem posed by the ancient Platonian computer. Proficiency with the Fourier transform and Bessel functions saves the day. On the most recent and twelfth visit to the Glia-6, the exploration of the abandoned colony was postponed because of an urgent need to calculate the diffusivity of a protein on neutrophil surfaces. The outcome of the calculation was needed in diagnosing medical problems with two midshipmen.

The details of these last six visits are available on the web at

http://www.me.rochester.edu/courses/ME201/webproj/proj.html

The present thirteenth visit to the Glia system was supposed to be centered on a continuing study of the ancient Platonian Museum of Science and Technology (shown in Figure 1). However, a protracted dust storm is keeping everyone on the Enterprise. The leisure activities of your group were summarily interrupted by Captain Kirk and Science Officer Spock. They give you extensive documentation on an ancient Platonian mini nuclear reactor. They then ask you to use that information to estimate how much external convective cooling is required to keep the maximum temperature in the reactor below a critical value.

Who is on the Enterprise?

Because of the excellent work done by your group as cadets – solving the puzzle of the acoustic lock and solving the mystery of the nuclear meltdown – Starfleet command took the unusual step of keeping you together as a research group after your graduation from the Academy. The wisdom of this decision was apparent the year before last, when your collective skills in thermal analysis allowed you to escape entombment in the Platonian Museum, and last year when you determined crucial protein diffusivities from FRAP data. In addition to you, the group members are Vinod, Catherine, Yud-ren, Roger, and Andi, all veterans of several trips, and Mamdouh, who joined the group last year. In honor of your group’s predilection for viewing life as a huge collection of boundary value problems, you are known collectively and informally as LambdaPro. For the present mission, LambdaPro has been traveling on the Enterprise, along with 291 crew members, all under the command of Captain Kirk, Science Officer Spock, Doctor McCoy and Chief Engineer Scott. You are waiting in orbit about Glia-6. When the dust storm subsides, your group will be part of a landing party sent down to continue the exploration of the abandoned colony. Meanwhile you have the thermal analysis of the mini reactor to contend with – a glorified homework problem, or so you all think.
THE PROBLEM: HOMEWORK IN SPACE

As our story opens, you are alone in the Bridge Deck Lounge, enjoying the view and an old-fashioned cup of earth-style coffee. As you are wondering about the rumor you have heard about a “homework” problem, your friend and fellow officer Catherine enters the lounge. She smiles and says, “I see you still drink that stuff that looks and tastes like dirt!”

“Earthy! The proper descriptor is earthy! And strong,” you reply. “Strong enough to dissolve those little pink umbrellas you like in your drinks!” You two have long disagreed on beverages, but manage to get occasional entertainment from the disagreement. Then frowning you ask, “What is this rumor about homework? Are we back in the Academy?”

“Interesting you should ask,” replies Catherine. “Roger asked me to find you and let you know that LambdaPro will be meeting at 1500 hours in the Junior Officer’s Lounge for a briefing on the assignment. It isn’t quite fair to call it a homework problem. It is really an interesting calculation on the control of the thermal state of an unusual mini nuclear reactor which the Platonians designed. We do have their detailed design manual which contains all of the parameters of their system.”

“I’m not convinced,” you reply. “If such a reactor no longer exists, and if there is no decision depending on the results of our calculations, it is just a homework problem in heat transfer, and we did more than enough of those in our Partial Differential Sciences course at the Academy. Prof. Clark had no end of long homework problems.”

“Maybe you’re right,” says Catherine, “but it is nearly 1500 hours, so let’s go to the meeting and get the details. Besides, you should leave the complaining to Andi. She does it much better than you!”

The two of you head off for the Junior Officer’s Lounge. When you enter the lounge, you look around and see the philosophical Mamdouh lost in thought, the history buff Vinod busy cleaning his prize 20th century blackboard (wherever does he get the chalk to keep it going?), your unofficial leader Roger pacing back and forth, impatient to start the meeting, and Yud-ren scribbling furiously, no doubt getting a jump start on the problem. Andi’s absence is not surprising, given that it is now only 1455 hours. Amazing and a little frightening, you think, that this eclectic, talented and prickly group now seems like family.

At 1457, Roger can wait no longer. “OK,” he shouts, “Listen up. We have a lot to do. We have exactly 48 hours to complete this analysis.” You think to yourself, what does it matter if it is right or wrong? It is just an exercise.

Just then Andi clatters into the room, tripping over a chair and ignoring Roger’s glare. She looks around. “What!? No cookies and coffee?” she complains, while collapsing clumsily and loudly into the chair that tripped her. “Couldn’t we have done this by email?” she continues.
Roger stares at her and finally says, “Are you through?” Andi snorts but says nothing more. Roger continues. “Vinod, will you please give us a detailed briefing on the calculation we are supposed to do?”

Vinod obligingly goes to his blackboard, picks up the chalk and begins writing and talking, 21st century professor style. “I’ll start by giving you a quick overview of the reactor and what we know about it. We can look at the technical details after that. The reactor core is spherical in shape and quite small. The power output is very small compared to typical 20th century earth reactors. It was evidently designed to provide power for remote and small settlements. In steady state, heat is generated by the fission reactions at a rate which varies with position in the core. Some of this heat is transferred to the reactor coolant, and the hot coolant is then used in the usual ways to generate electrical power. The remainder of the heat generated in the core is carried away from the outer surface of the core by forced convection. Our principal task will be to determine how much convection is needed to keep the maximum core temperature at a safe level.”

“Oh-ho,” you whisper to Catherine, “I think we are going to get one of those messy Newton’s law of cooling problems!”

“Even worse,” she whispers in reply, “it sounds like some kind of inverse problem!” By now Roger is glaring at you both, so you save further discussion for later.

Vinod continues. With a flourish, he picks up his chalk again and rapidly sketches the coordinates to be used. “We will be working in spherical coordinates,” he says. “The problem is axisymmetric, so we need only the radial coordinate $r$ and the polar angle $\phi$. The relevant steady-state quantities appearing in the heat balance are the reactor temperature $T(r, \phi)$, the average thermal conductivity of the reactor core $k$, the rate at which heat is added per unit volume per unit time by the fission reactions $\dot{q}(r, \phi)$, and the rate at which heat is removed by the coolant $\Gamma_c(r, \phi)$. Vinod continues talking while writing equations on the board. “The basic steady-state heat balance then gives the equation governing the temperature in the core,” he says as he writes equation (1) below. “Here $a$ is the radius of the core.” You are amazed at how fast he writes, and you think that it must have been tiring for 21st century students to sit through 50 minutes of this three or four times a week.
\[ k \nabla^2 T = -\Gamma_h + \Gamma_c, \quad 0 \leq r < a, \quad 0 \leq \phi \leq \pi, \]

where \[ \nabla^2 T = \frac{1}{r^2} \left( \frac{\partial}{\partial r} \left( r^2 \frac{\partial T}{\partial r} \right) + \frac{1}{\sin(\phi)} \frac{\partial}{\partial \phi} \left( \sin(\phi) \frac{\partial T}{\partial \phi} \right) \right). \] (1)

Vinod continues relentlessly. “The other major part of the problem is the boundary condition. Because the core surface is cooled by forced convection, we will use Newton’s law of cooling to describe the heat flow from the core surface to the air. Then the surface energy balance gives this boundary condition,” he says enthusiastically, as he writes ever faster.

\[ -k \frac{\partial T}{\partial r}(a,\phi) = h(T(a,\phi) - T_A). \] (2)

“Here \( h \) is the heat transfer coefficient and \( T_A \) is the ambient temperature.”

“I knew it,” you say to Catherine, “Newton’s law of cooling in the boundary condition.”

“Now,” says Vinod with even greater enthusiasm, “here is the really interesting part. We are given the parameters \( k, a, \) and the functions \( \Gamma_h(r,\phi) \) and \( \Gamma_c(r,\phi). \) We are also given the ambient temperature \( T_A \) and a critical temperature \( T_c. \) We must find the minimum \( h \) which keeps the entire reactor core below this critical temperature. The consequences of greatly exceeding \( T_c \) could be a melt-down, and the consequence of a modest excess of temperature over \( T_c \) could be eventual failure of a component because of thermal material damage.”

“I knew it!” Catherine says, “An inverse problem. That’s going to keep the computers busy.”

Roger, looking annoyed, says, “Do you two have a question?”

“Yes,” you shoot back, “how do you plan to solve this problem?”

This unexpected reply flusters Roger, whose forte is management rather than analysis. “I – I – I,” he stammers, “well, I –”.

Before he can continue, Vinod, always the peacemaker, says calmly and politely, “I think that is why we are all here. To determine together how we will solve this problem.” He is about to call on Yud-ren for some suggestions when Andi interrupts. “Expand everything in Legendre polynomials, and then solve the resulting ordinary differential equations for the \( r \)-dependent coefficients,” she says quickly and impatiently.

Mamdouh then joins the conversation. He speaks with precision in an unhurried way.

“The spherical harmonics are a complete set of functions for angular dependence. In this axisymmetric problem those harmonics reduce to the set of Legendre polynomials in \( \cos(\phi) \). The functions \( \Gamma_h \) and \( \Gamma_c, \) are known and hence will have known expansions. The unknown \( T \) may be expressed as a series in the Legendre polynomials with coefficients dependent on \( r \). We
may substitute these expansions into the partial differential equation and balance coefficients. This will give us an inhomogeneous ordinary differential equation for each such coefficient. We must find particular and homogeneous solutions for those equations, and we can use only the solutions which are regular at \( r = 0 \). Finally we must expand the boundary condition in those same harmonics, and thereby generate a boundary condition at \( r = a \) for each \( r \)-dependent coefficient in the series for \( T \).” He pauses for breath, which gives Andi a chance to break in.

“Didn’t I just say that?” she asks with some irritation. Mamdouh looks puzzled, as though he only now realized that Andi was even in the room. Meanwhile you are enjoying this interesting matchup between two very bright people, one quick and intuitive, the other systematic and philosophical.

Once again, Vinod to the rescue. “I think you both have outlined a good procedure for solving the problem,” he says gently. “Now. How do we deal with the fact that the heat transfer coefficient \( h \) is to be determined rather than having it given?”

There is a short silence, broken by Catherine. “The most elementary way would be by trial and error. We assume an \( h \), solve the problem, and find the maximum temperature. If the temperature we find exceeds \( T_c \), we make \( h \) larger and try again. If the maximum temperature we find is smaller than \( T_c \), we reduce \( h \) and repeat. Of course we could be more systematic and use the secant method on our iterates to find the next guess for \( h \). In any case, we will need the actual data for the reactor parameters before we can do such calculations.” Vinod responds to this by passing out copies of the data sheet (shown below in Table 1).

At this point Roger decides that it is safe to take over the discussion. “OK,” he says. “Here’s what we are going to do. We will form three groups and get three independent solutions to the problem. If the three groups agree, we can be fairly certain that their answers are right. I will be the liaison between the three groups and also with the senior officers. Mamdouh and Andi, I would like you to work together.” Andi sighs and Mamdouh smiles. Roger then turns to you and says, “You work with Catherine.” Finally he turns to Yud-ren and says, “Yud-ren I would like you to work with Vinod, but I am concerned. We haven’t heard anything from you.”

Yud-ren, who has been scribbling furiously throughout the meeting, says, “I have been working on the problem, and I already have the ordinary differential equations satisfied by the coefficients. I’m sure Vinod and I can finish this calculation today.”

Roger looks astonished at this, but you are not surprised. Yud-ren is very often a step or two ahead. You are most interested in how Andi and Mamdouh will work together. Will Mamdouh’s style become more pragmatic, or will Andi find a philosophical basis for her complaints? You find the people dynamics almost as interesting as the boundary value problems.

Once again Roger takes charge. “OK, let’s get going. We will meet here again tomorrow at 1500 hours to share our progress. Remember this has to be in final form by 800 hours the day after tomorrow.” Vinod and Yud-ren decide to continue working in the Junior Officer’s Lounge, rather than trying to make Yud-ren’s scattered work transportable. Andi waits
impatiently at the lounge door for Mamdouh, who is gathering his up his papers and his portable electronics slowly and carefully. You and Catherine head for the Bridge Lounge with its inspiring view. Roger, ever the Captain, is waiting to be the last to leave the Lounge.

Average core thermal conductivity \( k = 20.0 \text{ W/m} \cdot \text{K} \)

<table>
<thead>
<tr>
<th>Parameter</th>
<th>Value</th>
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</thead>
<tbody>
<tr>
<td>Core radius ( a )</td>
<td>1.5 m</td>
</tr>
<tr>
<td>Critical core temperature ( T_c )</td>
<td>720 °C</td>
</tr>
<tr>
<td>Temperature of air used in forced convection ( T_A )</td>
<td>20 °C</td>
</tr>
<tr>
<td>Heat production rate ( \Gamma_h(r,\phi) = a_h P_0(\cos\phi) + b_h r P_1(\cos\phi) + c_h r^2 P_2(\cos\phi) ) where ( a_h = 1.0 \times 10^5 \text{ W/m}^3, b_h = 0.9 \times 10^5 \text{ W/m}^4 ), and ( c_h = 4.8 \times 10^4 \text{ W/m}^5 )</td>
<td></td>
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<tr>
<td>Cooling rate ( \Gamma_c(r,\phi) = a_c P_0(\cos\phi) + b_c r P_1(\cos\phi) + c_c r^2 P_2(\cos\phi) ) where ( a_c = 0.7 \times 10^5 \text{ W/m}^3, b_c = 0.8 \times 10^5 \text{ W/m}^4 ), and ( c_c = 2.5 \times 10^4 \text{ W/m}^5 )</td>
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</table>

Table 1. Reactor Core Thermal Parameters

It is a day plus 6 hours later. You and Catherine are in the bridge lounge again, enjoying the thought of a job well done. The meeting of LambdaPro earlier in the afternoon had gone extremely well. The values of the heat transfer coefficient \( h \) obtained by the three groups differed by less than 2%, and you are all confident that the result is correct.

You are saying to Catherine, “It was fortunate that Vinod found those old notes from the Academy days – the notes on the solution of the Poisson equation in a sphere. Much of what we needed was there.”

“Yes,” she replies, “I am always amazed at how much stuff Vinod travels with, but this time it really paid off.”

“I think Yud-ren pretty much had the problem solved without the notes, but it was nice to have confirmation. How did you like it when Andi and Mamdouh gave separate presentations for their group work – like majority and minority opinions in a court ruling.”

“Yes,” says Catherine, “it was funny when Andi simply got up, stated the answer, and sat down. Mamdouh wore me down, though, with that history of spherical harmonics, and that definition of a very general class of problems that included this one as a special case. I timed him. It took him 23 minutes to get to the answer, and then he gave the same answer that Andi had given, but it was probably obtained by a much more rigorous method!”

“At least our homework is done,” you reply. Maybe we will even get a 5-point bonus for being on time! Oh! When is it that we give our report to the senior officers tomorrow?”
“At 800 hours,” replies Catherine. “Our results are so good that it should be a quick meeting. Now that the dust storm is waning, maybe we can make the visit to the planet surface.” Thinking back on this event much later, Catherine was reminded of the old saying that you should be careful what you wish for.

The next morning you all gather in the Junior Officers Lounge for the meeting, and even Andi is on time. Captain Kirk and Science Officer Spock stride briskly into the lounge at exactly 800 hours. You lean over to Catherine and whisper, “A fanatical punctuality is the hobgoblin of little minds.” Catherine looks a bit startled at the early morning philosophy, but says nothing.

Vinod, who is sitting directly behind you, smiles and says, “You are taking liberties with Emerson.” This strange conversation ends abruptly when you realize Kirk is glaring at you.

“Listen up!” Kirk commands. “Instead of hearing your reports on your calculations, we are going to let you test your results directly on the reactor.” You can hardly believe what you are hearing. Kirk continues. “Mr. Scott and his assistants have restored the reactor. They have also carefully calibrated the relation between the cooling air flow and the heat transfer coefficient, so that the reactor operator can set any value desired for the heat transfer coefficient. However the original Platonian manual had no information on the value needed for steady state. Thanks to your calculations, we now know that value.” He pauses. “We do know it, don’t we?” he asks with a sardonic smile. “We have left the honor of starting up the reactor to your group. We will leave immediately for the planet surface so that you can get started.”

Roger finally finds his voice. “But sir, I thought this was a homework exercise.”

“There is no homework in space,” Kirk snaps. “Everything counts. A wrong answer doesn’t mean 10 points off, it means that people might die!”

You all make the trip in silence to the transporter room on Morey deck. Just yesterday you were so confident that your answer was correct. Now it doesn’t seem nearly so certain. You will know in a few hours.
YOUR TASK AND WORK SCHEDULE

You should find the minimum value of the heat transfer coefficient $h$ for which the maximum core temperature just reaches the critical value $T_c$. Your answer should be supported by relevant calculations and graphs. Your results should include a contour plot of the temperature in the reactor core. You should explore how robust your answer is by looking at the maximum temperature for a few values of $h$ near your value.

In order to allow time for the evaluation of your work before term-end, and for any necessary cleanup of the reactor if your answer is wrong, your report must be completed by stardate 10140.0 (local Rochester time: Friday December 12 by 6 PM EST).

REFERENCES

The sketch on the cover is by Maggie Summers, an illustrative designer. The sketch is on the web at http://www.designerm.com/2005/2005_07_01_archive.html. The picture of the Enterprise was taken from the web: http://www.geocities.com/SiliconValley/Heights/4847/.

Figure 1 is a picture of the Sydney Opera House from the 1998 Grolier MultiMedia CD Encyclopedia.

Figure 2 shows reactor isotherms as created by Mathematica.

FIGURES

FIGURE 1. ANCIENT PLATONIAN MUSEUM OF SCIENCE AND TECHNOLOGY

FIGURE 2. TYPICAL ISOTHERMS IN PLATONIAN REACTOR