

THE MUSIC OF THE SPHERES

ME 201/MTH 281
PROJECT 2001



GENERAL DESCRIPTION OF PROJECT

In this project, you are on a training exercise for Starfleet Cadets. You are aboard the USS Enterprise, which is in the Glia starsystem orbiting the planet Glia-6. This is the first return of the Enterprise to Glia-6 since Captain Kirk's near-fatal encounter with a glion there on Stardate 8378.2. The training exercise will include a visit to the surface of Glia-6, as part of an ongoing research effort to determine the origins of the abandoned colony there. By an unlikely turn of events, you are presented with the opportunity to unlock the secrets of this lost culture, provided you can solve a certain problem in acoustics. The details of all this are given below.

PRACTICAL MATTERS

You may work alone, or in groups of two. You may discuss the project freely with others. If you get significant help from someone else, that fact should appear as an acknowledgment or reference in your report. Any books or articles that you use also should be referenced. Your report should be brief but self-contained, so that a reader not having access to these directions will know what you have done.

A typical report organization might be something like the following: Introduction (with a general description of the problem); Formulation (the detailed quantitative formulation of the problem); Results (results of the calculations, including any graphs or tables necessary to make the results clear); Summary (a summary of your conclusions); References (books, articles or people consulted). Although typed reports are preferred, neatly handwritten reports are acceptable. Your project grade will be based on both the write-up (20%) and the technical content (80%). For groups of two, only one report need be submitted.

DETAILED DESCRIPTION OF PROJECT PROBLEM

Background on the Glia System

Visits of the USS Enterprise to the Glia starsystem began on Stardate 5526.3, with an examination of an ecosystem on Glia-2 (ME163 F93). On that visit Starfleet Academy Cadets saved an ecosystem. They were commended even though their bold plan violated the Prime Directive. The second visit to the Glia system, on Stardate 6872.1, involved the launch of an unmanned probe to the large airless planet Glia-4 (ME 163 F94). Showing great ingenuity, the Cadets on that mission successfully designed a nonlinear landing suspension for the probe. The discovery on Glia-4 of naturally occurring dilithium crystals (used in warp propulsion) led to the third visit on stardate 7304.6, during which Academy Cadets solved a crucial forced vibrations problem, allowing the dilithium crystals to be transported undamaged to the Enterprise (ME163 S97). Because of the importance of the dilithium mining, a colony in a biodome was set up on Glia-4 to continue the mining operation. An epidemic of Jasmine fever on Glia-4 was the occasion of the fourth visit. Once again Academy Cadets proved the value of mathematics in modeling real events, and developed a vaccine transport strategy which minimized the number of lives lost (ME 163 S98). The fifth adventure in the Glia system (stardate 8378.2, ME163 S99) began innocently as a recreational visit to the abandoned colonies on Glia-6. The situation was instantly transformed into a dire emergency when Captain Kirk was bitten by a feral glion carrying a deadly virus. Only by the most strenuous efforts were the Academy Cadets able to save Kirk, by developing an optimal treatment protocol for the viral infection. The sixth training mission to the Glia starsystem (Stardate 8993.1, ME 163 S2000) required Cadets to calculate re-entry orbits for a 20th century Apollo capsule, and then, in a surprise test of their computational acumen, to actually ride the capsule down to the surface of Glia-3, accompanied by Professor Clark's cat Billie. Additional details on some of these earlier visits to the Glia system are available on the web at

<http://www.me.rochester.edu:8080/courses/ME163/webproj/proj.html>.

The seventh visit to the Glia system (Stardate 9604.2, ME201/MTH281) was the first involving upper level cadets from Starfleet Academy. On that visit, two cadets well-versed in partial differential equations were able to avert a war with the Klingons by their skillful analysis of a cooking problem. More details on that episode are available on the web at

<http://www.me.rochester.edu:8080/courses/ME201/F2000/webproj/proj2000.pdf>

The present eighth visit is the second visit of upper level cadets.

Who is on the Enterprise?

In addition to the 294 crew members, the Enterprise is carrying two Cadets from Star Fleet Academy, both with advanced training in partial differential sciences and you are one of those two Cadets. By chance, both you and the other Cadet have a strong interest in acoustics and music, and this turns out to be the key to deciphering the lost culture on Glia-6.

The Excursion to the Surface of Glia-6

Glia-6 is a class-M planet about 0.6 earth size with a thin but breathable atmosphere. The present visit is in mid-summer, as are all other visits to Glia-6, because the winters there are similar to those in upstate New York before the creation of the Federation Center for Climate Control. Glia-6 is most famous for its abandoned colony and the associated mystery of the origins and fate of the long-gone colonists. Glia-6 is also known because of Captain Kirk's close encounter a few earth years ago with a feral glion and Lana fever.

The landing party - consisting of you, your companion Cadet, Captain Kirk, Science Officer Spock, and several others - is wandering through the colonial ruins, which are known for their soaring forms and conspicuous lack of right angles (Figure 1). As the group wanders through the lower level of the ruins, Spock is explaining that it is not known where the colonists came from or what happened to them.

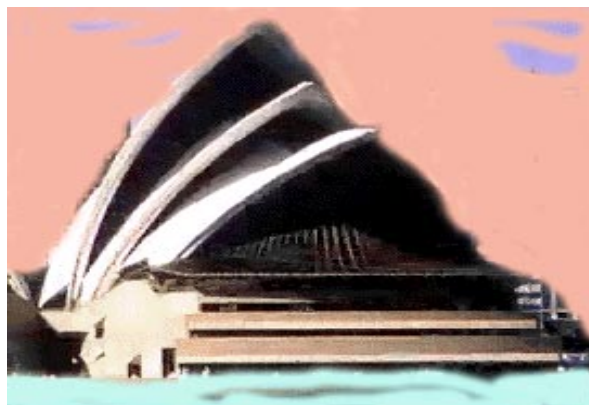


Figure 1. Late Colonial Ruins on Glia-6

As you walk, you ask Spock, “Is this all that remains of the colony, or are there other ruins elsewhere?”

Spock replies, “There are a few important sites elsewhere, but they are clearly thousands of earth years older than this site, and little remains of the original buildings. Curiously, the architecture was totally different in the early sites.” Spock proceeds to describe these earlier sites in some detail (Figure 2).



Figure 2. Early Colonial Ruins on Glia-6

As Spock delivers his precise, complete and logical description, you are amazed at what you hear, and you exclaim, “Those ruins sound like something from ancient Greece on earth!”

Spock grants your point, but says in an amused voice, “It hardly seems likely that there could be any connection with an ancient Earth civilization.”

You don’t agree, and in fact this all connects very well with something you have studied extensively. You are so excited you can hardly get the words out: “Don’t you see, this could be a colony of Platonians!”

Spock, in his thorough style, begins explaining to you what you already know. He says, “A small group of Platonians left their original planet many earth millennia ago, when they realized that their star Sahndara was going to go nova. They settled on Earth in ancient Greece at the time of Plato. When the glory days of the Greek culture faded, they moved on to another planet which they called Platonius, and they patterned their culture after the ancient Greek culture. But I am telling you things you surely must know already.”

“Yes but that is not the whole story!” you exclaim. “When they left ancient Greece, there were two space ships. One arrived at Platonius, and the other was never heard from again. It is just possible that the second ship landed here on Glia-6.”

“A fascinating conjecture,” says Spock. “But it would not be logical to accept it without proof. How is it that you have such extensive knowledge of the Platonians?”

“It is through my interest in music,” you reply. “As much as any civilization in the galaxy, they have developed both music and acoustics. They even have acoustic versions of everyday devices that for us would be mechanical – devices such as locks, for example.”

At this point Captain Kirk appears and says, somewhat condescendingly, “Well Cadet, are you taking advantage of this opportunity to learn about the Glia-6 mystery from Science Officer Spock?”

Before you can reply, Spock says, “It is somewhat the other way, Captain. This young Cadet believes that the colony on Glia-6 may have originated with a lost Platonian space ship.”

Kirk, making no effort to conceal the scorn in his voice, says to you, “Is that the kind of wild conjecture they teach in the Academy these days? When you are with Science Officer Spock, you would do well to listen more and talk less!”

You control your anger and say quietly, “It is a hypothesis which can be tested by looking for similarities between what is found here and what is found on Platonian. It is something to be decided by facts, and not opinions sir.”

Kirk, remembering his own days as a brash cadet, softens somewhat and even seems amused. He says, “I like your spirit cadet, but the truth is that almost nothing other than these ruins and the older ruins elsewhere remain of the colony that was here. We have nothing of their history, nothing of their culture, and nothing of their everyday life. But by all means look for yourself. We have another 4 hours here. You may succeed where all others have failed.”

On this sarcastic note, Kirk strides off, obviously with things to do which are much more important than talking with a Cadet. Just then, your companion Cadet joins you, and you introduce her to Spock, telling him that her knowledge of music, acoustics, and related mathematics is at least the equal of your own. Spock then asks the two of you if you would like to see the lowest level of the ruins. You eagerly agree and the three of you descend a narrow winding staircase to arrive in a room that is a perfect rectangular parallelepiped with very smooth walls. The dimensions are strange. It is about 20 m long, 8 m high but only about $\frac{2}{3}$ m in width – just room enough for one person to walk through. Suddenly a possibility occurs to you. You say to Spock, “Sir, what is the present temperature, and what is the composition of the atmosphere on this planet?”

Spock, amused, looks at his PDA/scanner and says, “The present temperature is 21.3 °C. The atmosphere, although thinner than earth’s, has an almost identical composition. And may I ask how you propose to use this information?”

You do some quick mental arithmetic before replying and then say, “According to my calculations, there is a resonant acoustic mode across the width of this hall that has a frequency almost exactly equal to middle C!”

Spock, still amused, replies, “You do indeed see the world in acoustical terms! And how should we use this newly discovered resonance?”

You turn to your companion cadet, among whose talents is perfect pitch, and ask her to sing a middle C as loudly as she can. She does so but nothing happens. You say to her, “Try again, but this time vary the pitch a little on either side of the middle C.” She does so, and when she reaches a few Hertz above middle C, the resonance kicks in sharply and the corridor reverberates dramatically. Suddenly there is a grinding noise from the far end of the corridor, and your companion stops singing abruptly. You all hurry single file to the end of the corridor, where you see that a panel has opened in the end wall, revealing a very smooth metal sphere, about 10 cm in diameter, and attached to the wall behind the panel.

Spock turns to you and says, “Very impressive, Cadet! How did you know this would happen?”

You reply, "I wasn't sure, but I knew that if this were truly a Platonian site, its secrets would be protected by various acoustic locks, and now I am sure that I am right! That sphere is an acoustic lock of the kind used by present day Platonians!"

Spock says, "So. You have opened one lock to find another. And how does this spherical lock work?"

After inspecting the top of the sphere, you answer, "It has a small opening in the top, into which one inserts an acoustic transducer. If the transducer is programmed to generate the correct frequencies, the lock will open."

"Fascinating," says Spock. "When the lock opens, what do you suppose we will find? And how does one know which frequencies to use?"

"My guess is that we will find a repository of historical material about the colony here," you say. "The colonists wanted to leave it for any Platonians eventually finding this site. As for the frequencies, they always think in terms of resonance. For the acoustic locks they presently use, one has to input, in order from lowest to highest, the first three nonzero resonant frequencies, maintaining each frequency for exactly 3 seconds. In the absence of any more specific information, that is probably what we should try."

Spock ponders a moment. Frowning, he says, "Suppose your guess is wrong. Do you have any idea what will happen when we activate the transducer with the wrong frequencies?"

Your companion answers promptly, "The Platonians are a very private people. In fact they kept their existence a secret from the Federation until their leader Parmen fell ill and needed medical assistance from the Enterprise. Because of this it is likely that the repository will self destruct if an attempt is made with incorrect frequencies."

Spock suggests that before making decisions, the three of you find out as much as you can about the sphere. Spock carries out a very low energy scan of the sphere, and learns that it has an outer diameter of 10.23 cm and a wall thickness of 0.87 cm. His scanner shows that the gas inside is pure Argon 40.

You are further excited by this result, and say, "That is consistent. The Platonians always use pure Argon 40 in their acoustic locks. The pressure won't matter as long as it is not too high, because the sound speed will then depend only on the temperature."

Your companion replies, "The temperature in this underground corridor cannot vary much. We can take it to be the 21.3 °C that was measured earlier."

Spock says that it is time to consult with Captain Kirk about what to do next, so the three of you hurry up the narrow stairway to the surface, hoping that the newly opened panel will stay open, or, if it closes, that you will be able to open it again.

When you find Kirk, Spock relates all that has happened. Kirk, who rarely has any doubts about anything, says immediately that this is a wonderful opportunity to solve a great mystery, and that the lock must be opened as soon as possible. You reply that you and your companion know how to calculate the resonant frequencies and that you will start on that right away.

Kirk says, "That should be easy. Just find the frequencies of the first three radially symmetric modes."

Your companion replies sharply, “With all due respect sir, I don’t think that the modes with the lowest three frequencies will all be radially symmetric. In fact I am quite sure that they won’t be.”

Kirk turns to Spock and says, “Tell me, where is the Academy finding this new breed of feisty Cadet!?” Then he turns to you and your companion and says, “Just make sure that your calculations are correct!”

Spock muses for a moment and then says to your companion, “Tell me Cadet, what should we call this little adventure?”

Your companion thinks for a moment and then replies, “Why not call it the Music of the Spheres?”

Spock replies, “A good choice, although probably not what Pythagoras had in mind.”

“Nor Kepler,” you add. Spock responds with a slight appreciative smile. Kirk looks slightly puzzled, shrugs, and strides away, no doubt on important business again.



Figure 3. The acoustic lock on the historical repository of GliA-6. The unlocking transducer is inserted into a small opening on the top.

The Acoustic Equations

As shown in the Academy course Applied Partial Differential Science, the pressure disturbance P in an acoustic wave satisfies the wave equation:

$$\frac{\partial^2 P}{\partial t^2} = c^2 \nabla^2 P . \quad (1)$$

Here c is the sound speed in the medium supporting the wave. For a standing wave of frequency ω , the solution has the form

$$P(\mathbf{r}, t) = \cos(\omega t) \Psi(\mathbf{r}) . \quad (2)$$

Substitution of the form (2) into equation (1) leads to the Helmholtz equation for the spatial amplitude $\Psi(\mathbf{r})$:

$$\nabla^2\Psi = -k^2\Psi, \text{ where } k^2 = \frac{\omega^2}{c^2}. \quad (3)$$

As shown at the Academy, the acoustic boundary condition on a rigid surface S is

$$\mathbf{n} \cdot \nabla\Psi(\mathbf{r})|_S = 0, \quad (4)$$

where \mathbf{n} is the unit normal to the surface. Equations (3) and (4) constitute an eigenvalue problem whose complete solution will give all of the frequencies of standing modes.

In the case of a spherical cavity of radius a , we use spherical coordinates r , ϕ , and θ , as defined in class and as shown in Figure 4. Then equations (3) and (4) become

$$\frac{1}{r^2} \frac{\partial}{\partial r} \left(r^2 \frac{\partial\Psi}{\partial r} \right) + \frac{1}{r^2 \sin\phi} \frac{\partial}{\partial\phi} \left(\sin\phi \frac{\partial\Psi}{\partial\phi} \right) + \frac{1}{r^2 \sin^2\phi} \frac{\partial^2\Psi}{\partial\theta^2} = -k^2\Psi, \text{ with } \left. \frac{\partial\Psi}{\partial r} \right|_{r=a} = 0. \quad (5)$$

Fortunately a simplification is possible here. It turns out that although there are modes with θ -dependence, each such mode has the same frequency as a mode without θ -dependence. Thus in looking for the frequencies of the standing modes, we may limit attention to solutions of the form

$$\Psi = \Psi(r, \phi). \quad (6)$$

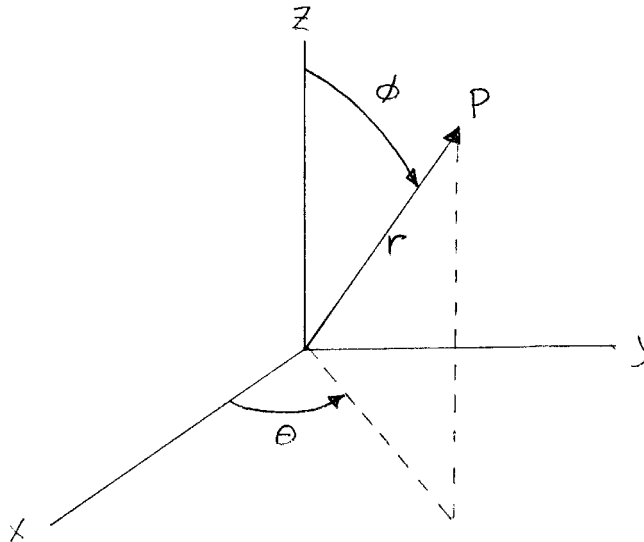


Figure 4. Spherical Coordinate System Used for Helmholtz Equation

Useful Information on Solution Technique

In assignment #10 you showed that solutions (6) of equation (5) could be found in the separated form

$$\Psi(r, \phi) = F_n(r)P_n(\cos \phi), \quad (7)$$

where P_n is the n th Legendre polynomial and where the radial function $F_n(r)$ satisfies

$$\frac{d}{dr} \left(r^2 \frac{dF_n}{dr} \right) - n(n+1)F_n + k^2 r^2 F_n = 0. \quad (8)$$

In this same assignment, you also found the solutions for F_0 and F_1 . In this project, you will need to treat the radial equation more generally. You will do this by connecting the radial functions with Bessel functions, by showing that the substitution given below leads to Bessel's equation for $G_n(r)$. This will help greatly in your computation of the frequencies of the modes, because Bessel functions are part of the standard kernel in Mathematica.

$$F_n(r) = \frac{G_n(r)}{\sqrt{r}}. \quad (9)$$

The Parameter Values

We summarize here all of the parameter values and other relevant information for the problem.

Outside diameter of the sphere: 10.23 cm.

Wall thickness of the sphere: 0.87 cm.

Gas in the sphere: pure Argon 40.

Temperature of gas in the sphere: 21.3 °C.

Pressure of gas in sphere: unknown but estimated to be about one Earth atmosphere.

Your Task

You must find the lowest three nonzero modal frequencies by completing the solution outlined above.

Work Schedule

In order to allow time for the evaluation of your work before term-end, your report on the acoustic lock must be completed by stardate 9881.7 (local Rochester time: Wednesday December 12 by 6 PM EST). If you are successful, you will go down in Federation History as a member of the group which unlocked the secrets of the Glia-6 culture. If however your calculated frequencies are wrong, the repository will self-destruct when the attempt is made to open the lock, and the knowledge of Glia-6 culture will be lost forever. If that happens, you will have to begin considering other career choices.

Examples of Pressure Distribution for Several Modes

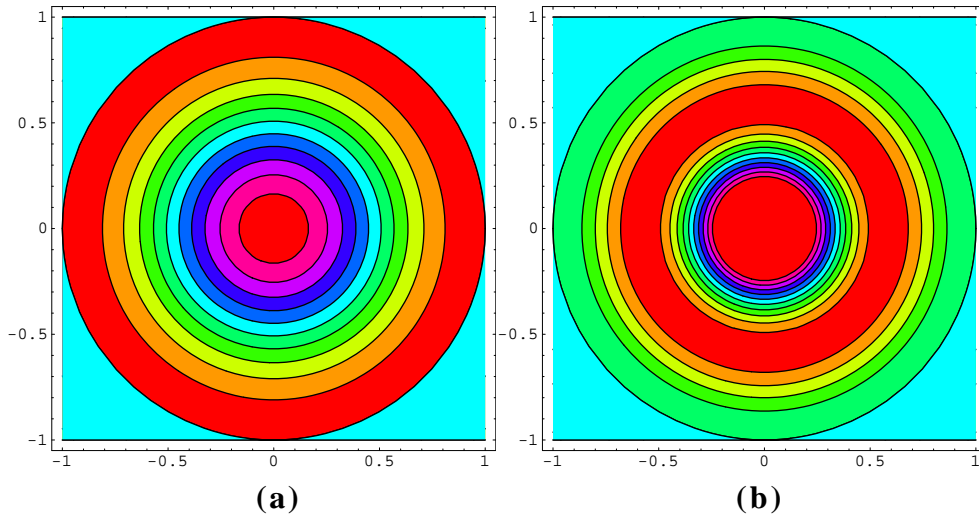


Figure 5. Contours of constant pressure for radially symmetric modes: (a) first radial mode and (b) second radial mode.

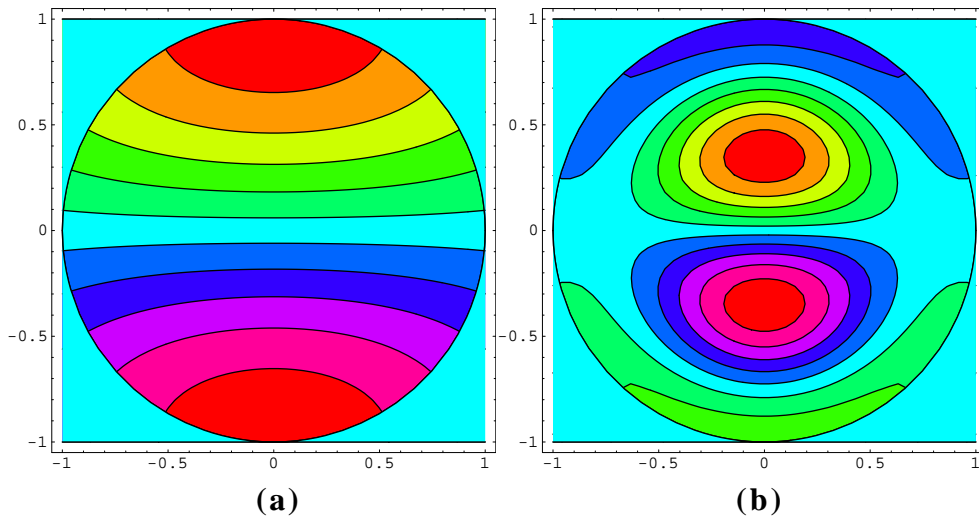


Figure 6. Contours of constant pressure for modes with angular dependence $P_1(\cos\phi)$: (a) first radial mode and (b) second radial mode.

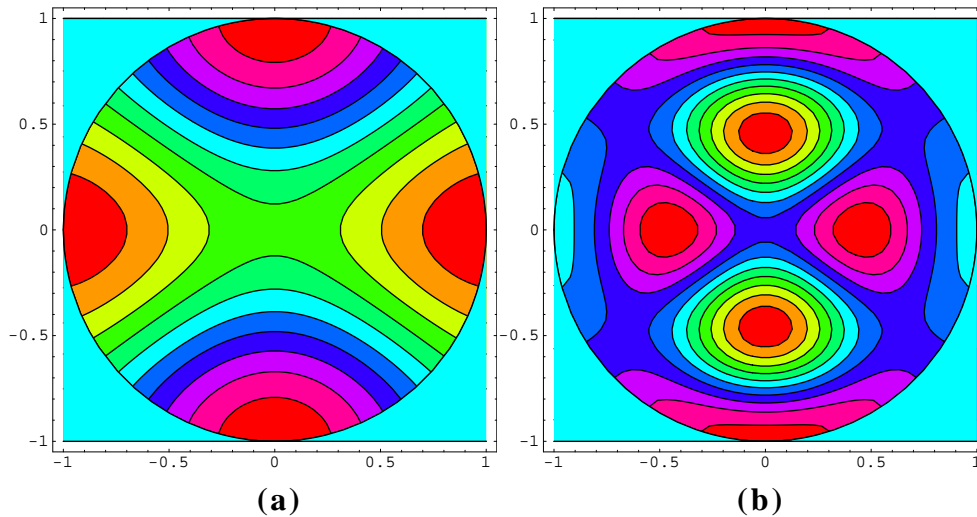


Figure 7. Contours of constant pressure for modes with angular dependence $P_2(\cos\phi)$: (a) first radial mode and (b) second radial mode.

Cover picture from **Mysterium Cosmographicum**, by Johannes Kepler, 1596, as reprinted in **Cosmos** by Carl Sagan, p. 58, Random House, 1980. Figures 1 and 2 from the 1998 Grolier MultiMedia CD Encyclopedia. Figure 3 courtesy of Professor P.A. Clark of RIT. Figures 5, 6, and 7 prepared using ContourPlot in Mathematica.

The Platonians appeared in the Stark Trek Episode “Plato’s Stepchildren.” Information on this and other aspects of Star Trek from **The Star Trek Encyclopedia**, Michael Okuda, Denise Okuda, and Debbie Mirek, Pocket Books, 1994.