7.6 Spherically Symmetric Transient Heat Conduction in a Sphere with Sources and with a Newton’s Law of Cooling Boundary Condition

**A. formulation**

In this lecture we develop the basic mathematical theory for the boundary value problem associated with the project. Your task in the project will be to translate this theory into numbers, results, and conclusions.

We analyze the temperature $T$ in a sphere of radius $a$, when the temperature is a function only of the spherical radial coordinate $r$ and the time $t$: $T = T(r, t)$. In the sphere there is a volume heat source $\Gamma(r, t)$ (SI units W/m$^3$). The equation governing $T$ is then the energy balance we derived earlier in the course:

$$\rho c \frac{\partial T}{\partial t} = \frac{1}{r^2} \frac{\partial}{\partial r} \left( r^2 \frac{\partial T}{\partial r} \right) + \Gamma,$$  \hspace{1cm} (1)

where we have already assumed that the thermal conductivity $k$ is constant, and where we will assume that density $\rho$ and specific heat per unit mass $c$ are both constant. We divide the equation by $\rho c$ to get

$$\frac{\partial T}{\partial t} = \frac{1}{\rho c} \frac{1}{r^2} \frac{\partial}{\partial r} \left( r^2 \frac{\partial T}{\partial r} \right) + \frac{\Gamma}{\rho c},$$  \hspace{1cm} (2)

where

$$D = \frac{k}{\rho c}$$  \hspace{1cm} (3)

is the constant thermal diffusivity (m$^2$/s).
And

\[ T(r, t) = \frac{\Gamma(r, t)}{\pi C} \left( \frac{K}{r^2} \right). \]  

(4)

We take the initial temperature to be \( f(r) \), a given function of \( r \):

\[ T(r, 0) = f(r). \]  

(5)

The boundary condition on the surface of the sphere is described by Newton's law of cooling with a constant ambient temperature \( T_a \):

\[ -k \frac{dT}{dr}(r, t) = h \left( T(r, t) - T_a \right). \]  

(6)

In most problems involving the Laplace operator and spherical geometry, we have to deal with Legendre functions or spherical Bessel functions or both. Fortunately for us, this problem with purely radial variation, is an exception. A simple change of dependent variable gives us an easier problem. We let

\[ T(r, t) - T_a = \frac{U(r, t)}{T}. \]  

(7)

It is straightforward to substitute this into the above equations defining the problem. The result is
\[ \frac{\partial \psi}{\partial t} = D \frac{\partial^2 \psi}{\partial r^2} + r \psi, \quad 0 < r < a, \quad t > 0 \quad (8) \]

\[ \psi(0, t) = 0 \quad (9) \]

\[ -k \frac{\partial \psi}{\partial r}(a, t) = (h - \frac{K}{a}) \psi(a, t), \quad (10) \]

\[ \text{and IC:} \quad \psi(\xi, 0) = r(T(r, 0) - T_0). \quad (11) \]

The boundary condition (9) was not given, but it is essential. We see from (7) that if this condition is not satisfied, \( T \) will be singular at \( r = 0 \). For convenience in later calculations, we rescale the boundary condition (10):

\[ \psi(a, t) = -\sigma a \frac{\partial \psi}{\partial r}(a, t), \quad (12) \]

where

\[ \sigma = \frac{1}{\text{Bi} - 1} \quad (13) \]

with \( \text{Bi} = \frac{h a}{K} \).

The quantity \( \text{Bi} \) is the Biot number. Let the limit as \( \text{Bi} \to \infty \), the boundary temperature will be equal to the ambient \( T_0 \), and hence in this limit, \( \psi = 0 \) at \( a \). The condition (12) reduces to that for \( \sigma \to 0 \) which is equivalent to \( \text{Bi} \to \infty \).
B. Appropriate Eigenfunctions

We are going to solve this problem by expansion of \( \psi \) and \( r \xi \) in a set of appropriate eigenfunctions. "Appropriate" means that the eigenfunctions are eigenfunctions of \( \frac{\partial^2}{\partial r^2} \), and that they satisfy the same homogeneous boundary conditions as \( \psi \). Thus our eigenfunctions will be generated by the Sturm–Liouville system

\[
\frac{d^2 \phi}{dr^2} = -\lambda \phi, \quad 0 < r < a \quad (15)
\]

\[
\phi(0) = 0, \quad \phi(a) = -\sigma a \frac{d\phi}{dr}(a). \quad (16)
\]

It is easy to show that the Rayleigh Quotient is

\[
\lambda = \frac{\int_0^a \phi'^2 \, dr + \sigma a \phi(a)^2}{\int_0^a \phi^2 \, dr}, \quad (17)
\]

and from here we can show that \( \lambda > \delta \).

We solve the problem now. The general solution of the equation for \( \phi \)

\[
\phi = A \cos(\sqrt{\lambda} r) + B \sin(\sqrt{\lambda} r). \quad (18)
\]

The condition \( \phi(0) = 0 \) tells us that \( A = 0 \). We impose the second boundary condition (and take \( B = 1 \)) to get

\[
\sin(\sqrt{\lambda} a) = -\sigma \sqrt{\lambda} a \cos(\sqrt{\lambda} a). \quad (19)
\]

We let

\[
\beta = \sqrt{\lambda} a. \quad (20)
\]
The eigenvalue equation is then
\[ \lambda n(z) = -\sigma z. \quad (21) \]
\[ -\sigma n(z) \]

The graph shows the countable infinity of intersections and roots \( z_n \). Then the \( n^{th} \) eigenfunction and eigenvalue are
\[ \lambda_n = \frac{z_n^2}{a^2}, \quad \phi_n(r) = \sin\left(\frac{z_n r}{a}\right), \quad \eta = 1, 2, 3, \ldots \quad (22) \]

We may use these functions to expand a given function \( f(r) \). The coefficients are obtained as usual by orthogonality:
\[ f(r) = \sum_{\eta=1}^{\infty} f_\eta \phi_\eta(r), \quad \text{where} \]
\[ f_\eta = \frac{\int_0^a f(r) \phi_\eta(r) \, dr}{N_\eta} \quad (23) \]
where \( N_\eta = \int_0^a \phi_\eta^2(r) \, dr = \frac{a}{2} \left[ 1 - \frac{\sin(2z_n)}{2z_n} \right] \).
C. Solution of the Boundary Value Problem

We now solve the problem for \( \psi(r, t) \) and hence for \( T(r, t) = T_0 + \frac{\psi(r, t)}{r} \). For convenience we repeat the formulation here.

\[
\begin{align*}
\text{Eq:} & \quad \frac{\partial \psi}{\partial t} = \Delta \psi + r \frac{\partial \psi}{\partial r}, \text{ } 0 < r < a, t > 0 \\
\text{BC:} & \quad \psi(0, t) = 0, \quad \psi(a, t) = -a \frac{\partial \psi}{\partial r}(a, t), \quad (24) \\
\text{IC:} & \quad \psi(r, 0) = r(T(r, 0) - T_0). 
\end{align*}
\]

We expand both \( \psi \) and \( rT \) in a series of the eigenfunctions \( \phi_n(r) \):

\[
\psi(r, t) = \sum_{n=0}^{\infty} \psi_n(t) \phi_n(r), \quad rT(r, t) = \sum_{n=0}^{\infty} \phi_n(t) \phi_n(r),
\]

where \( \psi_n(t) = \frac{1}{N_n} \int_0^a \phi_n(r) \psi(r, t) \, dr \).

For \( t=0 \), \( \psi(r, t) \) must reduce to the initial condition, so

\[
\psi(r, 0) = r(T(r, 0) - T_0) = \sum_{n=0}^{\infty} \psi_n(0) \phi_n(r),
\]

hence \( \psi_n(0) = \frac{1}{N_n} \int_0^a (T(r, 0) - T_0) \phi_n(r) \, dr \).

We substitute the expansions (25) into the first of equations (24), and then we determine the coefficients. We get

\[
\frac{d\psi_n}{dt} + \delta_n^0 \psi_n = \phi_n(t)
\]

where \( \delta_n^0 = \frac{D2}{\sigma^2} \frac{a}{G^2} \).
We use an integrating factor to get
\[
L_n(t) = C_0(t) e^{-\rho_0 t} + \int_0^t d_0(t') e^{-\rho_0 (t-t')} dt',
\]
where \( C_0(t) \) is given by (26).

Then
\[
F(x, t) = \tilde{T}_0 + \frac{1}{\nu} \sum_{n=0}^{\infty} C_n(t) \phi_n(x)
\]
(29)