ME 201/MTH 281/ME400/CHE400 Solution of Laplace Equation by Convolution Integral: An Example *Mathematica* 7

1. Introduction

In this notebook we consider the solution of the boundary value problem given below for the Laplace equation in a two-dimensional upper half-space. The method is the Fourier transform and convolution.

$$\frac{\partial^2 \Phi}{\partial x^2} + \frac{\partial^2 \Phi}{\partial y^2} = 0 \;, \quad -\infty < x < \infty,\; y > 0\;, \qquad \text{with } \Phi(x, 0) = f(x), \text{ and } \Phi \underset{y\to\infty}{\longrightarrow} 0\,. \tag{1}$$

We obtained the solution to this problem in class by the Fourier transform, defined by

$$\tilde{\Phi}(k,y) = \int_{-\infty}^{\infty} \Phi(x, y)\, e^{-ikx}\, dx\,. \tag{2}$$

By taking the Fourier transform of the equation and boundary condition, we find the solution in the form

$$\tilde{\Phi}(k,y) = \tilde{f}(k)\, e^{-|k|\, y}, \tag{3}$$

where $\tilde{f}(k)$ is the Fourier transform of the boundary function $f(x)$. To invert this, we use convolution, along with the known inverse transforms

$$F^{-1}\left\{\tilde{f}(k)\right\} = f(x) \quad \text{and} \quad F^{-1}\left\{e^{-|k|\, y}\right\} = \frac{1}{\pi}\frac{y}{x^2 + y^2}. \tag{4}$$

Then the solution is

$$\Phi(x, y) = \int_{-\infty}^{\infty} \frac{1}{\pi} \frac{y}{(x - x')^2 + y^2} f(x')\, dx'\,. \tag{5}$$

2. Example

We choose a boundary potential that is constant on an interval (-*a*, *a)* and zero otherwise. We let the constant value be Φ0. Then the convolution solution is

$$\Phi[x\_, y\_] = \text{Simplify}\left[\Phi 0 \int_{-a}^{a} \frac{1}{\pi} \frac{y}{(x - x')^2 + y^2} \, dx' \text{ , Assumptions} \to \{a > 0, y > 0, x \in \text{Reals}\}\right]$$

$$\frac{\Phi 0 \left(\text{ArcTan}\left[\frac{a-x}{y}\right] + \text{ArcTan}\left[\frac{a+x}{y}\right]\right)}{\pi}$$

Now we choose values for *a* and Φ0 and then make a contour plot of the solution. We stay slightly away from *y* = 0 to avoid problems from the *y* in the denominator of the arguments of the ArcTan.

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a = 1.0 (** m **); Φ0 = 10.0 (** volts **);
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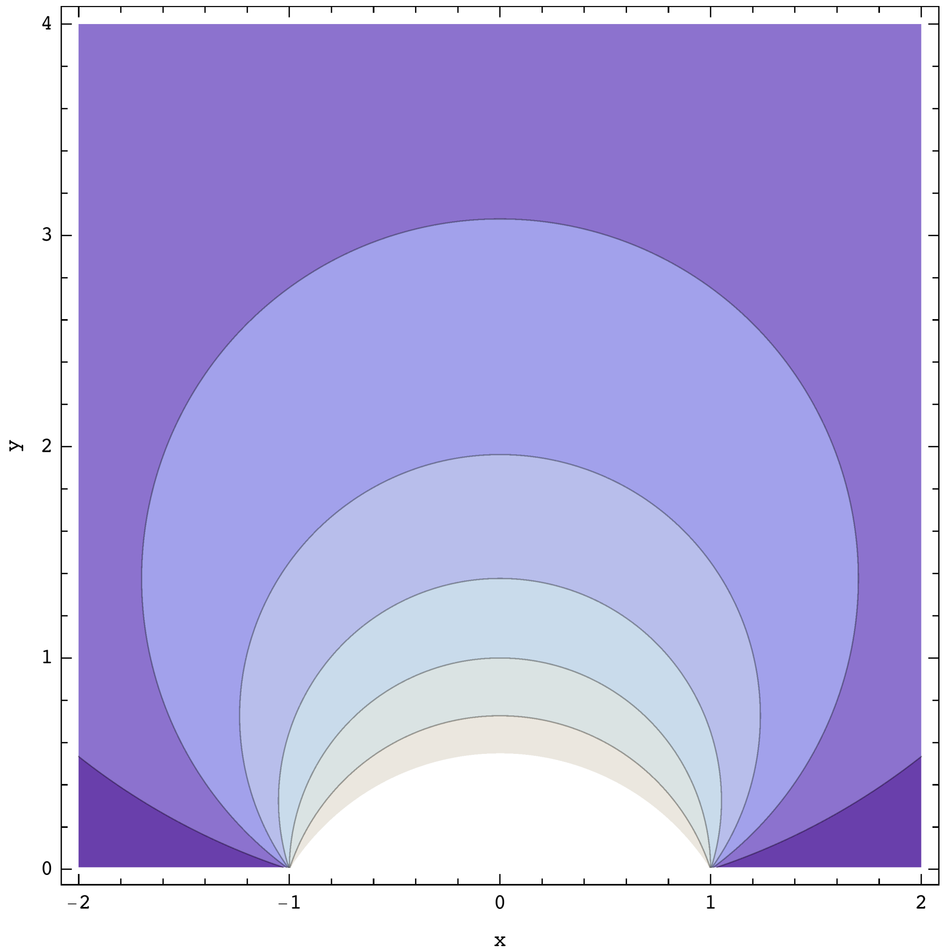
ContourPlot[Φ[x, y], {x, -2 a, 2 a}, {y, 0.01`, 4 a},
 PlotPoints → 200, FrameLabel → {"x", "y"}]
```



The equipotentials all reach the boundary at the discontinuities at $x = \pm a$, as we would expect. The contours look like circles. As an exercise, you might want to show analytically that they are circles.