

# ME 201/MTH 281/ME400/CHE400 Irreversibility in Heat Flow *Mathematica 7*

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## ■ 1. Introduction

In this notebook, we look at several solutions of the heat equation which illustrate irreversibility. The irreversibility shows up as a gradual loss of information about the initial state of the system as time progresses. We consider two examples of this. In the first example, given in section 2, we see three very different initial conditions, all approaching the same asymptotic state as time proceeds. In the second example, given in section 3, we see how rapidly the small scale components in the initial state will disappear. The problem solved in both of these examples is the initial value problem given below.

$$\frac{\partial T}{\partial t} = D_f \frac{\partial^2 T}{\partial x^2}, T(x, 0) = f(x), T(0, t) = 0, \text{ and } T(L, t) = 0. \quad (1)$$

The solution obtained in class by separation of variables is

$$T(x,t) = \sum_{n=1}^{\infty} B_n e^{-n^2 \pi^2 D_f t / L^2} \sin(n\pi x / L), \text{ where } B_n = \frac{2}{L} \int_0^L f(x) \sin(n\pi x / L) dx. \quad (2)$$

The nature of the irreversibility is clearly illustrated by the solution (2). At each time, the solution is a Fourier sine series. At the initial time, the coefficients are  $B_n$ , the Fourier sine coefficients of the initial condition. As time progresses, each coefficient decays to zero, but at different rates. The large  $n$  coefficients decay very rapidly. Thus information about the high-frequency components in the initial condition disappears first. As time progresses, more and more of the coefficients become unobservably small. For large times (typically  $t$  larger than the diffusion time  $L^2/(\pi^2 D_f)$ ), the solution is approximated by the first term in the series:

$$T(x,t) \approx B_1 e^{-\pi^2 D_f t / L^2} \sin(\pi x / L). \quad (3)$$

In this time regime, the only information surviving about the initial condition is  $B_1$ . Hence solutions with the same value of  $B_1$  will look the same at this time. Eventually  $T$  decays to zero and even this information about the initial condition is lost. The two examples presented below give graphic illustrations of this process.

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## ■ 2. Asymptotic Convergence of Solutions for Different Initial Conditions

In this section, we look at three different solutions for different initial conditions, chosen so that  $B_1$  is the same for each. We will construct a movie showing the evolution of the systems in time, and we will see all three solutions approaching the same asymptotic solution, given by equation (3).

As in class, we consider an aluminum slab of thickness 0.1 m, with thermal diffusivity  $70 \times 10^{-6} \text{m}^2/\text{s}$ .

$$\mathbf{Df} = 70.0 * 10^{-6}; (** \text{ m}^2/\text{s} **)$$

$$\mathbf{L} = 0.1 (** \text{ m} **);$$

For this set of parameters, the basic diffusion time in seconds is

$$\mathbf{tau} = \frac{\mathbf{L}^2}{\pi^2 \mathbf{Df}}$$

$$14.4745$$

This is the time over which considerable cooling of the slab takes place. The asymptotic solution (3) is generally valid for times comparable with or exceeding this. We take the exact solution to be the partial sums of the series (2) up to and including  $n = 20$ . The exponential factor in the 20th term drops to  $10^{-4}$  for  $e^{-n^2 \pi^2 D_f t / L^2} < 10^{-4}$ , hence for  $t$  exceeding

$$\frac{4 * \mathbf{Log}[10.] * \mathbf{L}^2}{20^2 * \pi^2 * \mathbf{Df}}$$

$$0.333287$$

So our representation of the exact solution should be valid for times of roughly 1/3 of a second or larger. In constructing the solutions, it is convenient to have an expression for the  $n$ th term apart from the coefficient  $B_n$ .

$$\mathbf{term}[\mathbf{x}_-, \mathbf{t}_-, \mathbf{n}_-] := e^{-n^2 \pi^2 D_f t / L^2} \mathbf{Sin}[\mathbf{n} * \pi * \mathbf{x} / \mathbf{L}]$$

We now define three different solutions corresponding to three different initial conditions. We choose the solutions to have the same value of  $B_1$ . The first solution corresponds to a constant initial condition, which we take to be  $T_1 = 100$  °C. We call the coefficients  $B_1[n]$ . By the formula in equation (2), we find that

$$\mathbf{T1} = 100;$$

$$\mathbf{B1}[\mathbf{n}_-] := \mathbf{If}[\mathbf{EvenQ}[\mathbf{n}], 0, 4 * \mathbf{T1} / (\pi * \mathbf{n})]$$

The value of the first coefficient is

$$\mathbf{B1}[1]$$

$$\frac{400}{\pi}$$

The solution corresponding to this we call Temp1.

$$\mathbf{Temp1}[\mathbf{x}_-, \mathbf{t}_-] := \mathbf{N}[\mathbf{Sum}[\mathbf{B1}[\mathbf{n}] * \mathbf{term}[\mathbf{x}, \mathbf{t}, \mathbf{n}], \{\mathbf{n}, 1, 20\}]]$$

The second solution is taken to be a positive constant  $T_2$  in the half interval  $[0, L/2]$ , and zero in the half interval  $[L/2, L]$ . By an easy calculation one shows that the Fourier sine coefficients  $B_2$  are

$$\mathbf{B2}[\mathbf{n}_-] := (2 * \mathbf{T2} / (\mathbf{n} * \pi)) * (1 - \mathbf{Cos}[\mathbf{n} * \pi / 2])$$

The first coefficient is

$$\mathbf{B2}[1] = \frac{2 T2}{\pi}$$

To match  $\mathbf{B1}[1]$  we choose

$$\mathbf{T2} = 200;$$

Then the second solution  $\text{Temp2}$  is

$$\mathbf{Temp2}[x_, t_] := \mathbf{N}[\mathbf{Sum}[\mathbf{B2}[n] * \mathbf{term}[x, t, n], \{n, 1, 20\}]]$$

For the third solution, we take an initial condition proportional to  $\sin(\pi x/L)$ . The coefficient of proportionality we call  $\mathbf{T3}$ . Thus

$$\mathbf{Temp3}[x_, t_] := \mathbf{N}[\mathbf{T3} * \mathbf{term}[x, t, 1]]$$

To match  $\mathbf{B1}[1]$  and  $\mathbf{B2}[1]$  we take

$$\mathbf{T3} = 400 / \pi;$$

Only the first coefficient is nonzero for this solution:

$$\mathbf{B3}[1] = \mathbf{T3}; \mathbf{B3}[n_] := 0;$$

We also define the initial conditions for *Mathematica* so that they can be plotted.

$$\mathbf{init1}[x_] := \mathbf{T1}$$

$$\mathbf{init2}[x_] := \mathbf{If}[x < L / 2, \mathbf{T2}, 0]$$

$$\mathbf{init3}[x_] := \mathbf{T3} * \mathbf{Sin}[\pi * x / L]$$

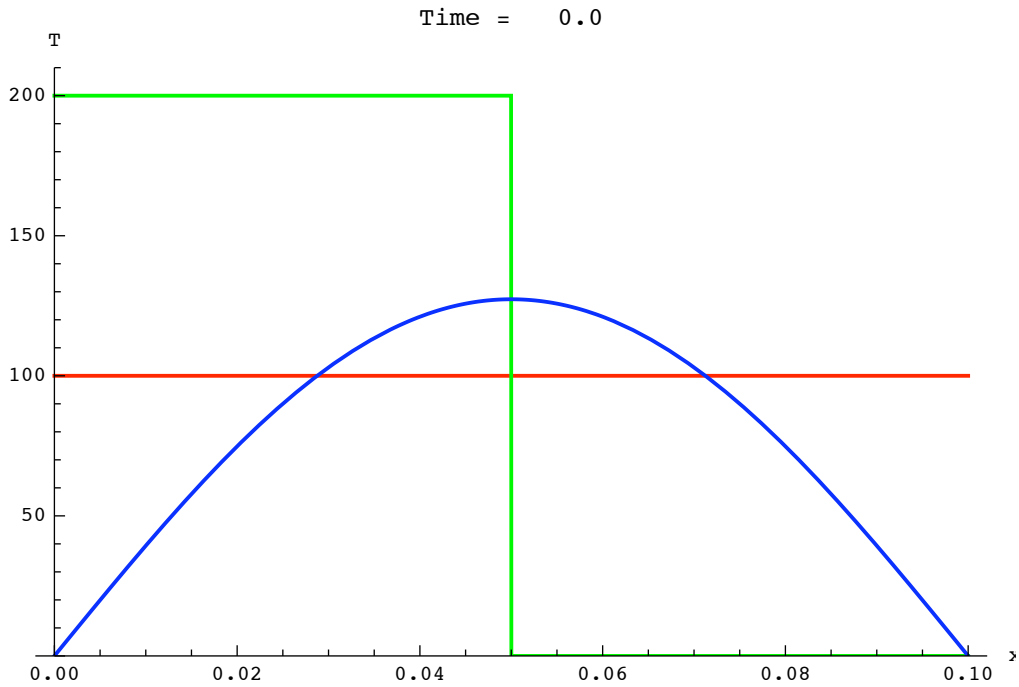
Now we define the functions that produce the graphs of the three solutions. We plot the first solution in red, the second in green, and the third in blue.

```
graph[0] := (t = 0; Plot[{init1[x], init2[x], init3[x]}, {x, 0, L}, PlotStyle ->
{{RGBColor[1, 0, 0], Thickness[0.004]}, {RGBColor[0, 1, 0], Thickness[0.004]},
{RGBColor[0, 0, 1], Thickness[0.004]}},
PlotRange -> {0, 2.1 * T1}, AxesLabel -> {"x", "T"}, ImageSize -> 380,
PlotLabel -> Row[{"Time =", PaddedForm[t, {4, 1}]}]])
```

```
graph[t_] := Plot[{Temp1[x, t], Temp2[x, t], Temp3[x, t]}, {x, 0, L},
PlotStyle ->
{{RGBColor[1, 0, 0], Thickness[0.004]}, {RGBColor[0, 1, 0], Thickness[0.004]},
{RGBColor[0, 0, 1], Thickness[0.004]}},
PlotRange -> {0, 2.1 * T1}, AxesLabel -> {"x", "T"}, ImageSize -> 380,
PlotLabel -> Row[{"Time =", PaddedForm[t, {4, 1}]}]])
```

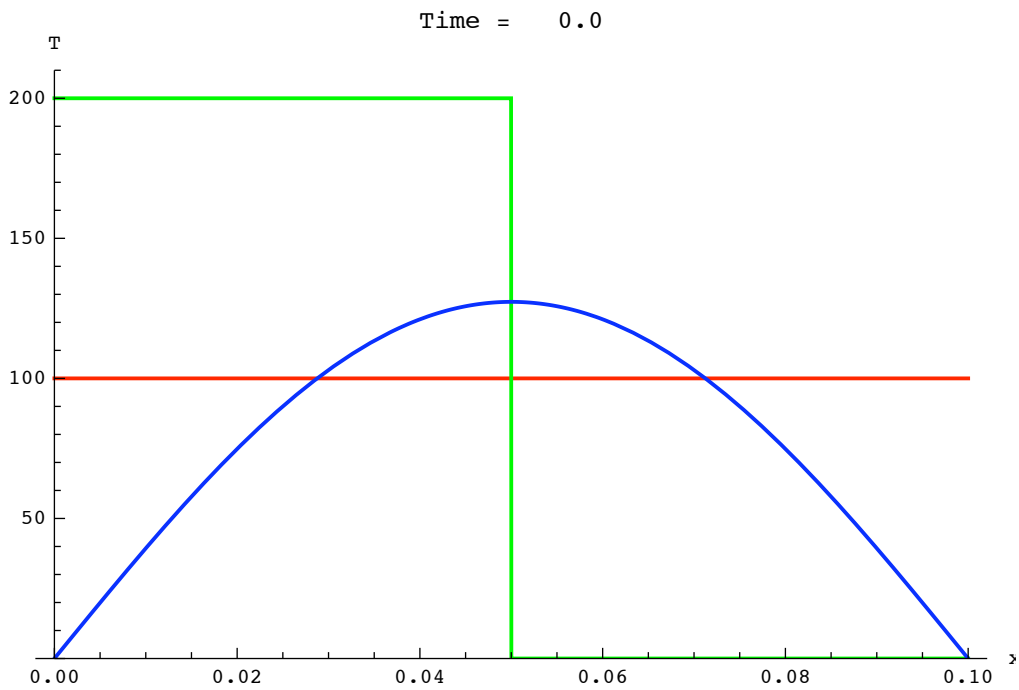
We construct the movie. We let time vary from 0 to 20 seconds in increments of 0.2 seconds, giving 101 frames altogether in the movie. In the printed version of this notebook, only the first graph of the sequence is shown.

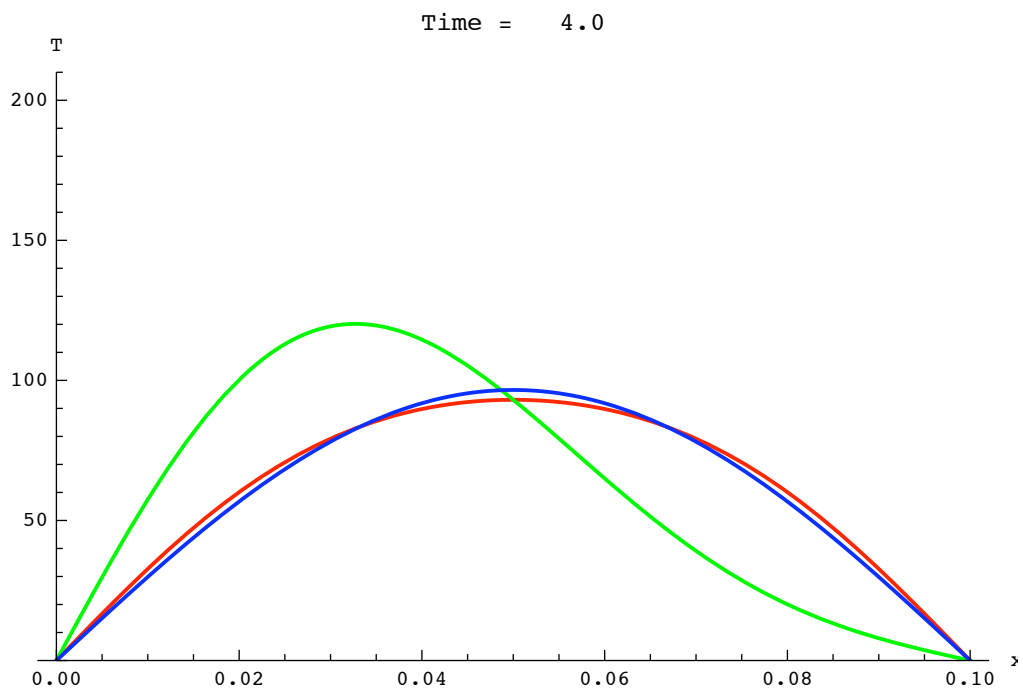
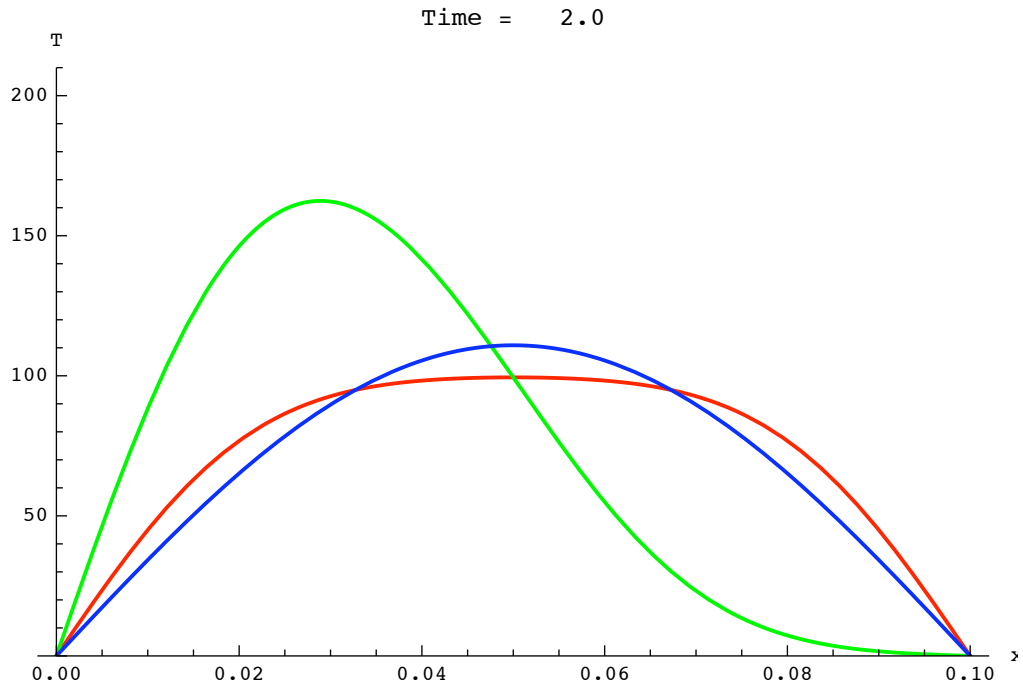
```
Do[Print[graph[0.2 * i]], {i, 0, 100}];
```

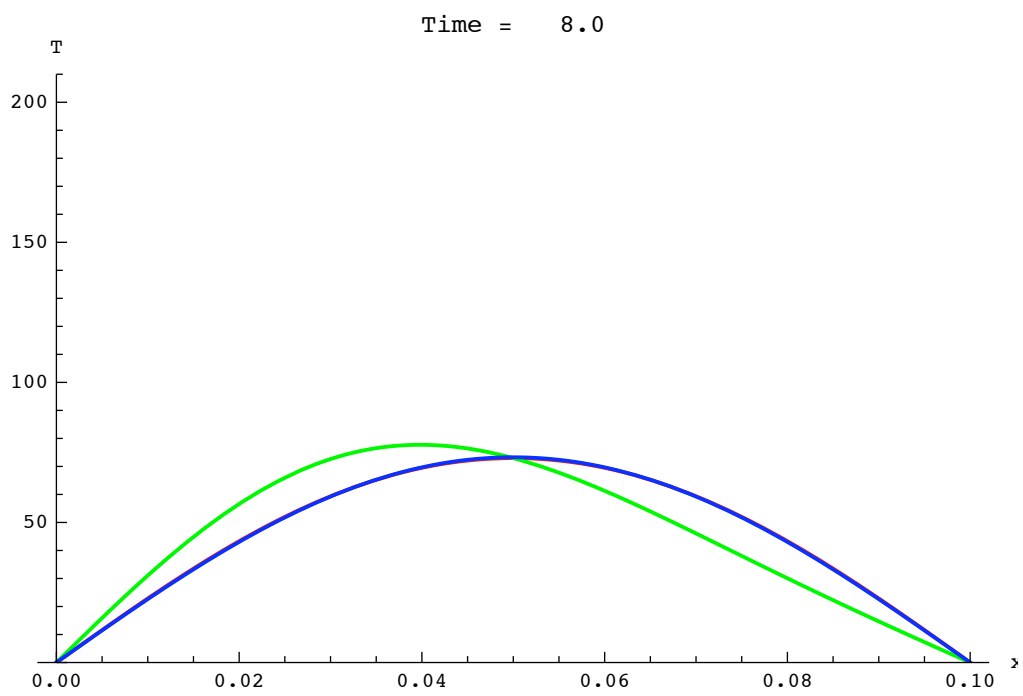
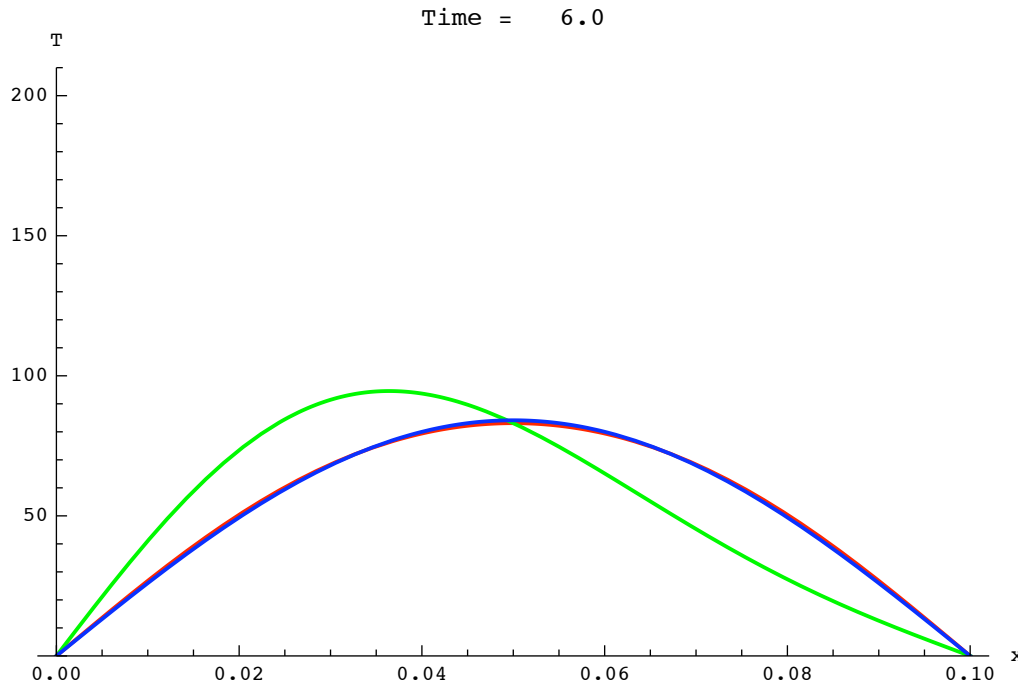


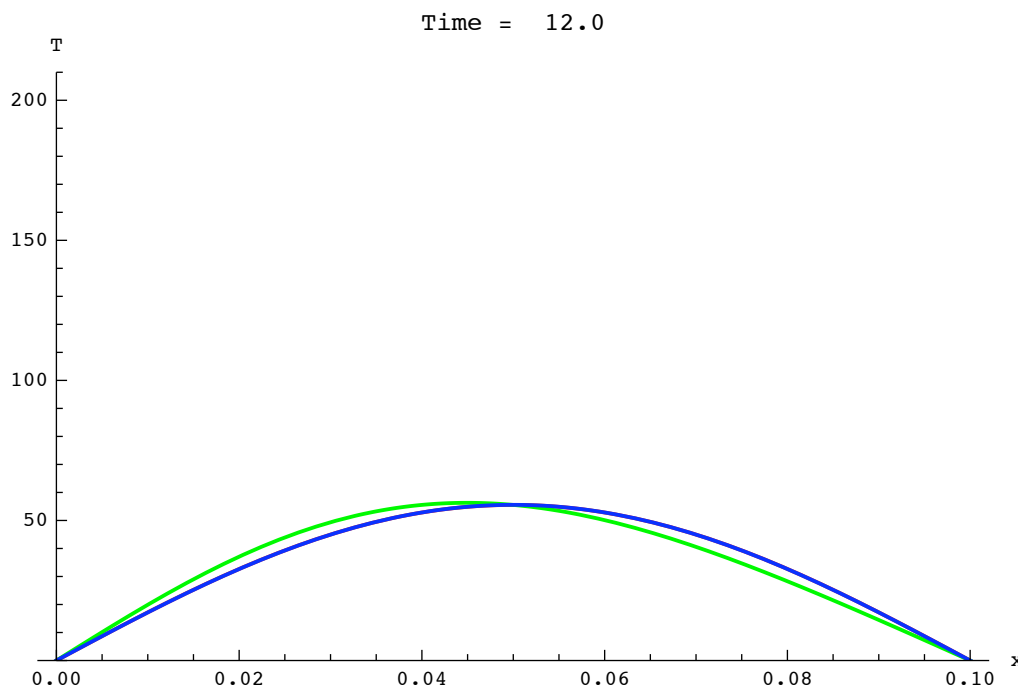
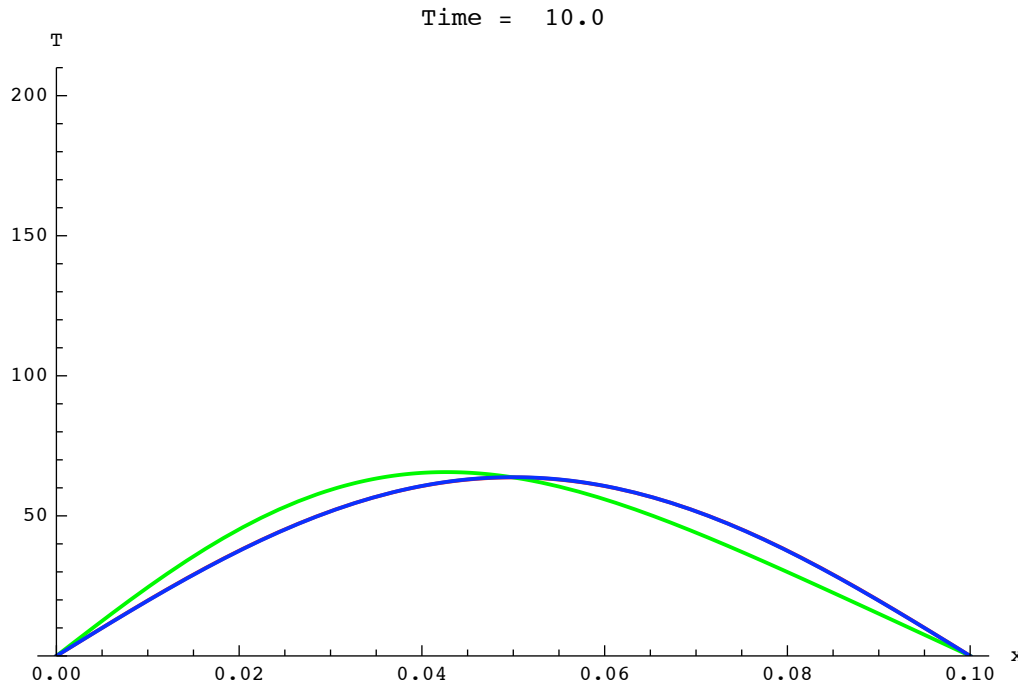
You can animate the graph sequence by selecting all of the graphs and then choosing the menu option **Graphics -> Rendering -> Animate Selected Graphics**. For the printed version of the notebook we construct an abbreviated sequence running from  $t = 0$  to  $t = 20$  in increments of 2 s.

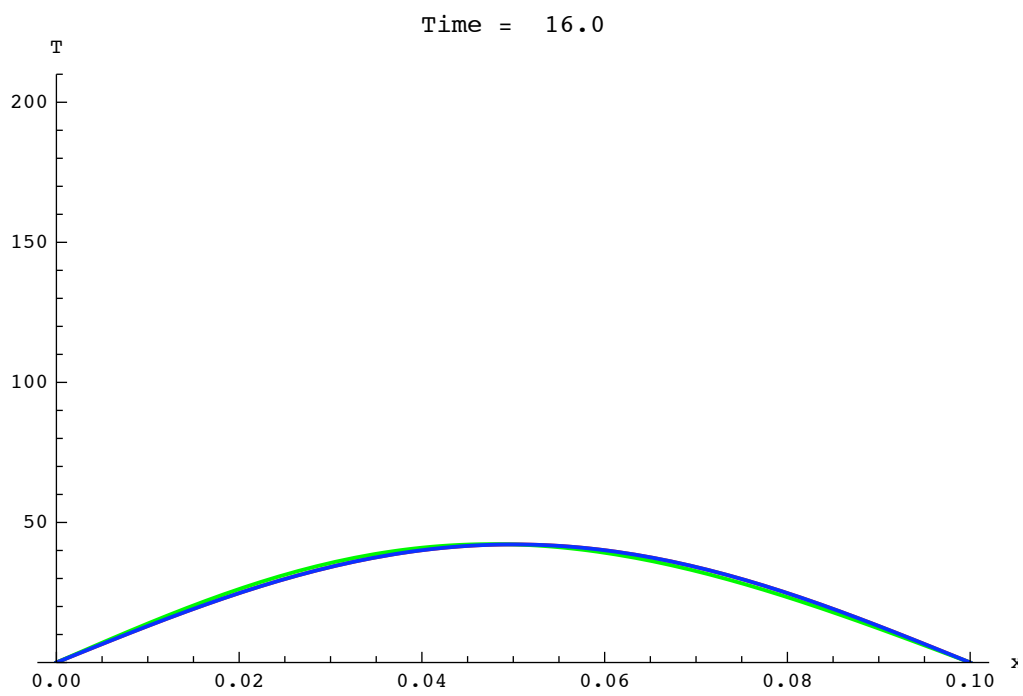
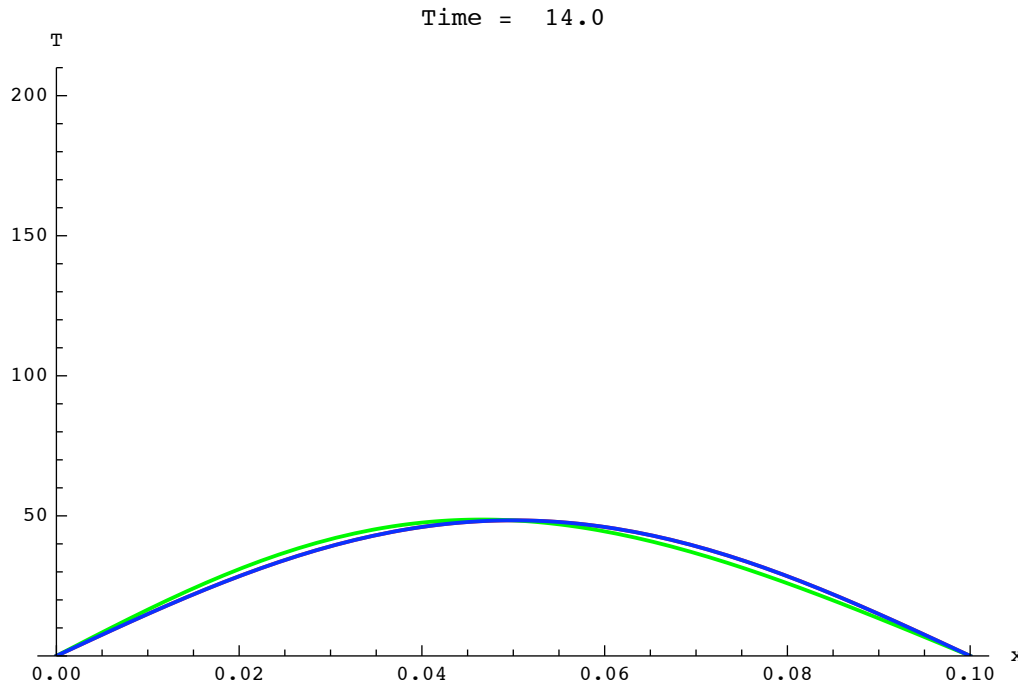
```
Do[Print[graph[2 * i]], {i, 0, 10}];
```

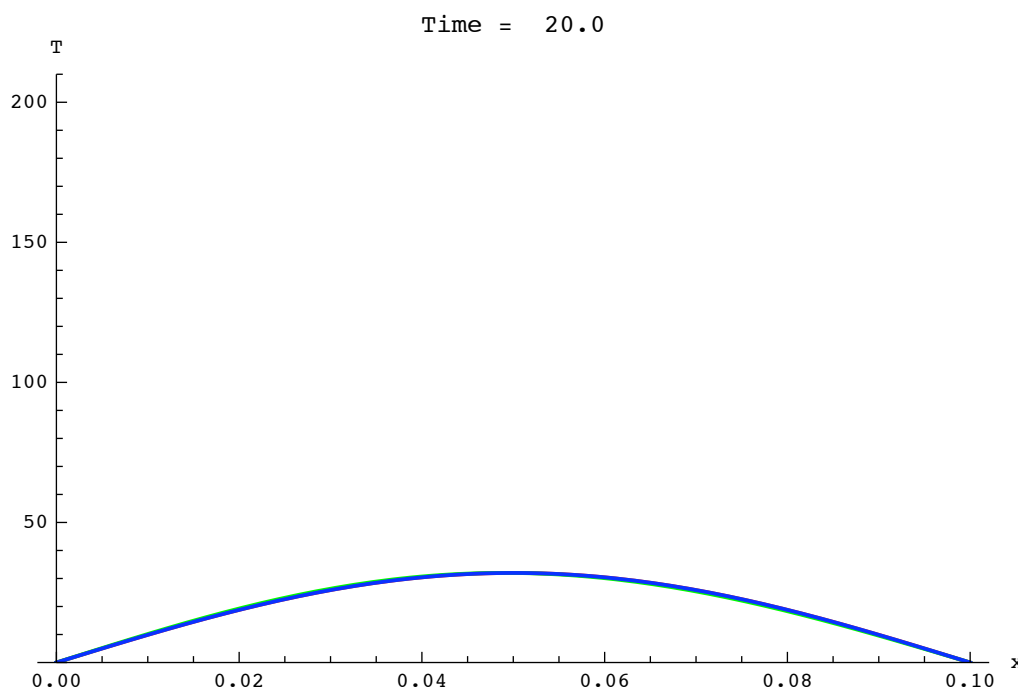
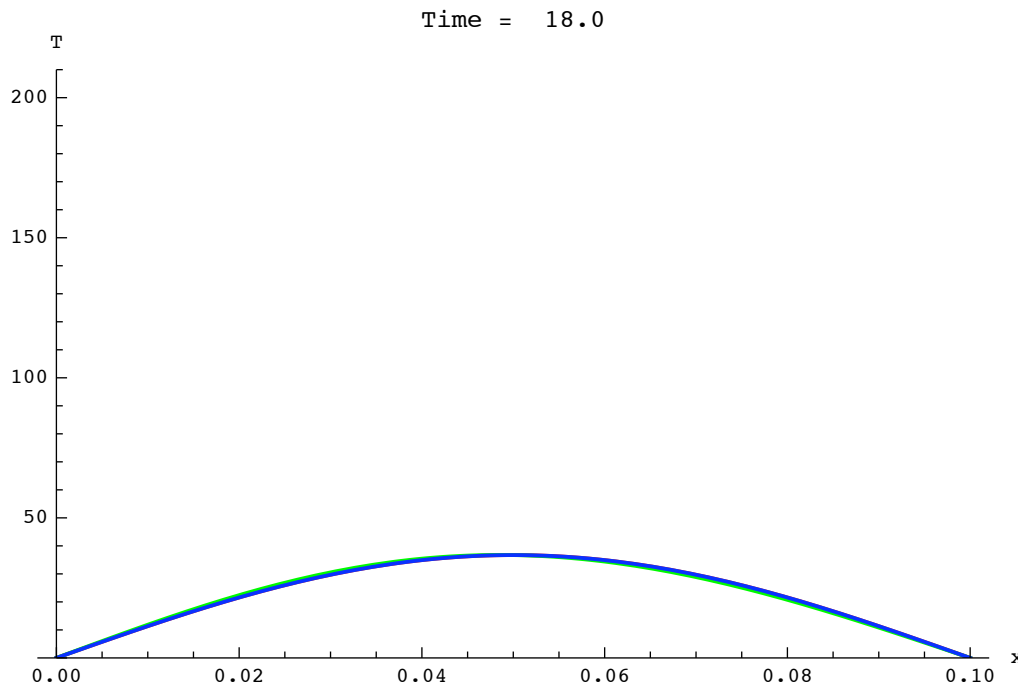












Notice that the red and blue profiles coalesce before the green one joins them. We can easily understand this if we look at the first three coefficients in each solution.

```
Print["B1[1] = ", B1[1], "   B1[2] = ", B1[2], "   B1[3] =", B1[3]]
```

$$B1[1] = \frac{400}{\pi} \quad B1[2] = 0 \quad B1[3] = \frac{400}{3\pi}$$

```
Print["B2[1] = ", B2[1], "   B2[2] = ", B2[2], "   B2[3] =", B2[3]]
```

$$B2[1] = \frac{400}{\pi} \quad B2[2] = \frac{400}{\pi} \quad B2[3] = \frac{400}{3\pi}$$

```
Print["B3[1] = ", B3[1], "   B3[2] = ", B3[2], "   B3[3] =", B3[3]]
```

$$B3[1] = \frac{400}{\pi} \quad B3[2] = 0 \quad B3[3] = 0$$

All solutions have the same first term. That is why they coalesce as time increases. The first and third solutions have a zero second term, whereas the second (green) solution has a non-zero second term. Because this second term in the green solution decays more slowly than third terms in the other solutions, it causes the green solution to deviate from the first term for a longer time.

### ■ 3. Rapid Decay of Initial Small Scales

In this section, we examine the rapid decay of small scale variations in the initial condition. This could for example represent small scale noise in the initial data. As a basic initial condition, we take the first harmonic,  $\sin(\pi x/L)$ . We give it an amplitude of  $T_0$ , where we take  $T_0 = 100$ . We retain the earlier values of slabwidth  $L$  and diffusivity  $D_f$ .

```
T0 = 100;
```

```
initbase[x_] := T0 * Sin[(π * x) / L]
```

We superpose on this a harmonic of order  $k$ , with amplitude a fraction  $f$  of the fundamental. We call the initial increment `initnoise[x]`.

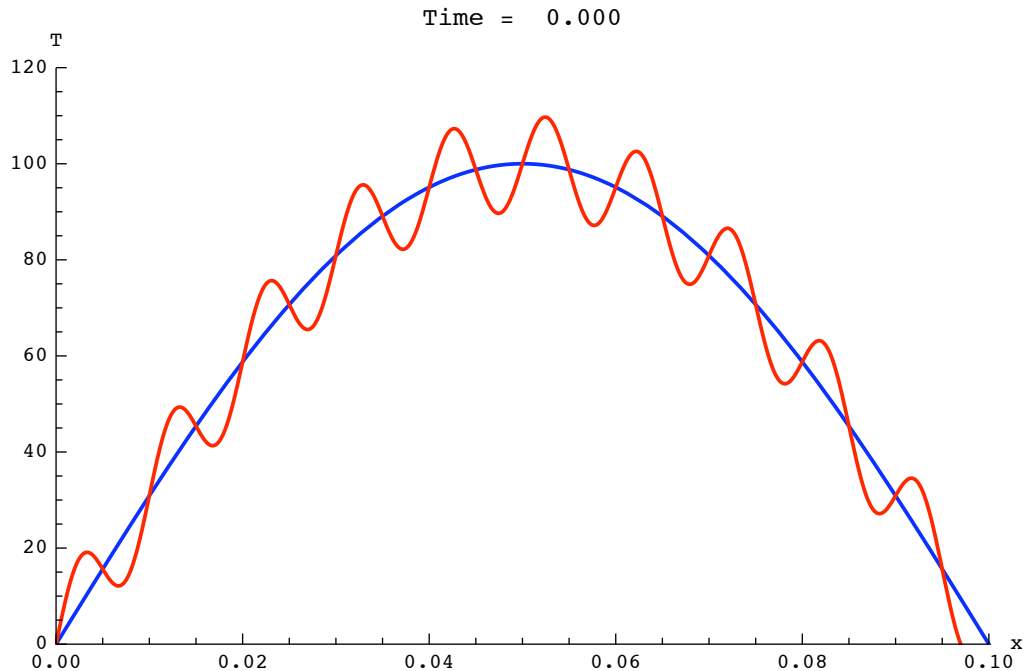
```
initnoise[x_] := f * T0 * Sin[(π * k * x) / L]
```

We choose  $k = 20$ , and we take the noise amplitude fraction  $f$  to be 0.1-- the noise is 10% of the base.

```
k = 20; f = 0.1;
```

We plot two initial conditions -- the basic initial condition in blue, and the initial condition with noise in red.

```
Plot[{initbase[x], initbase[x]+initnoise[x]}, {x, 0, L}, PlotStyle->
{{RGBColor[0, 0, 1], Thickness[0.004]}, {RGBColor[1, 0, 0], Thickness[0.004]}},
PlotRange->{{0.0, L}, {0, 1.2*T0}}, ImageSize->380, AxesLabel-> {"x", "T"},
PlotLabel -> Row[{"Time =", PaddedForm[0, {5, 3}]}]]
```



We are going to evolve both of these solutions in time to see what happens. Before doing that, let's calculate the diffusion times for the two harmonics. For the basic mode,

$$\mathbf{taubase} = \frac{\mathbf{L}^2}{\pi^2 \mathbf{Df}}$$

14.4745

For the noise harmonic,

$$\mathbf{taunoise} = \mathbf{taubase} / (\mathbf{k} * \mathbf{k})$$

0.0361861

The outcome is already clear from these numbers. The noise is not going to last much longer than a few multiples of 0.036 s -- maybe 0.1 s at best. The basic mode, however, will retain appreciable amplitude for times comparable with 14.5 s. Let's construct the graph sequence to get a picture of this process. We will plot the solution at time increments of 0.005 s for times running from 0 to 1 s. The basic solution is in blue, the solution plus noise in red.

$$\mathbf{basesol}[\mathbf{x}_-, \mathbf{t}_-] := \mathbf{initbase}[\mathbf{x}] * \mathbf{Exp}[-(\mathbf{Df} * \pi^2 * \mathbf{t}) / \mathbf{L}^2]$$

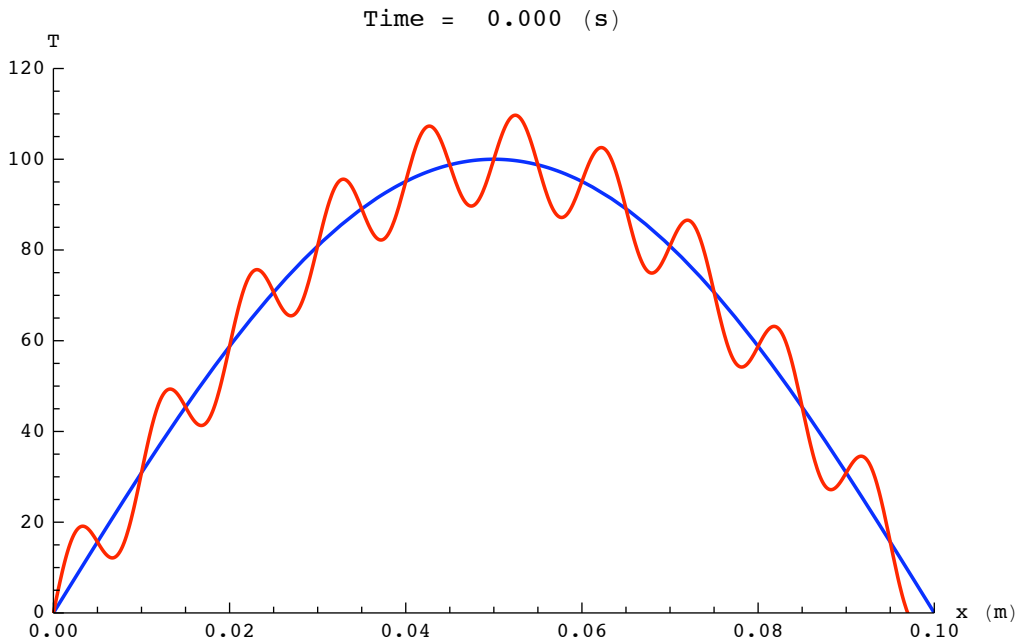
```
noiseinc[x_, t_] := initnoise[x] * Exp[- (Df *  $\pi^2$  * k2 * t) / L2]
```

```
noisesol[x_, t_] := basesol[x, t] + noiseinc[x, t]
```

First we define a function which graphs these two solutions versus  $x$  at a fixed time, and then we construct the graph sequence with a Do loop.

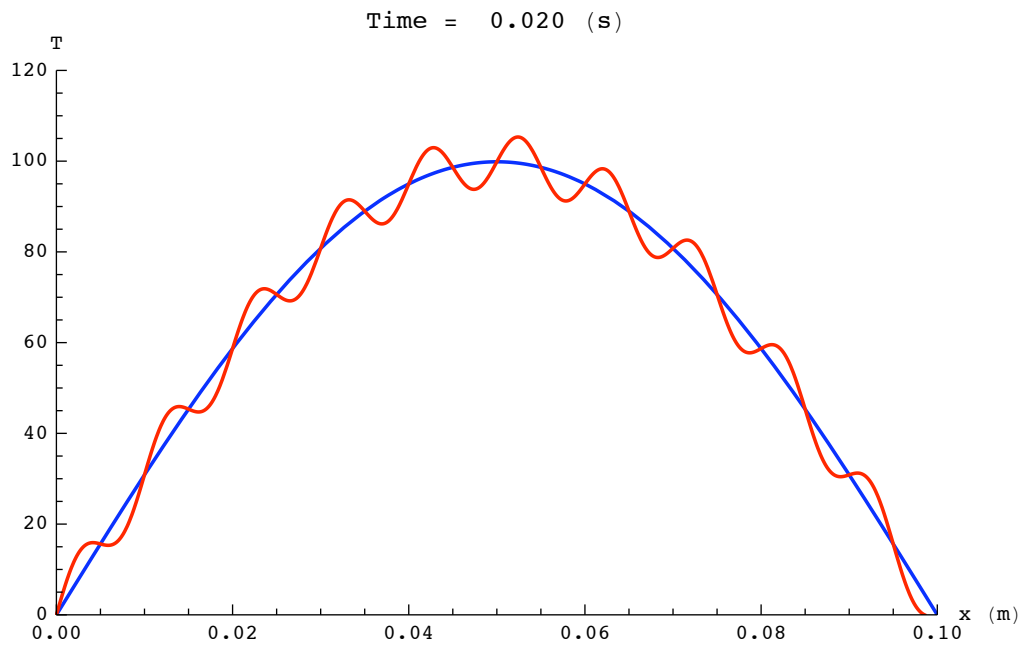
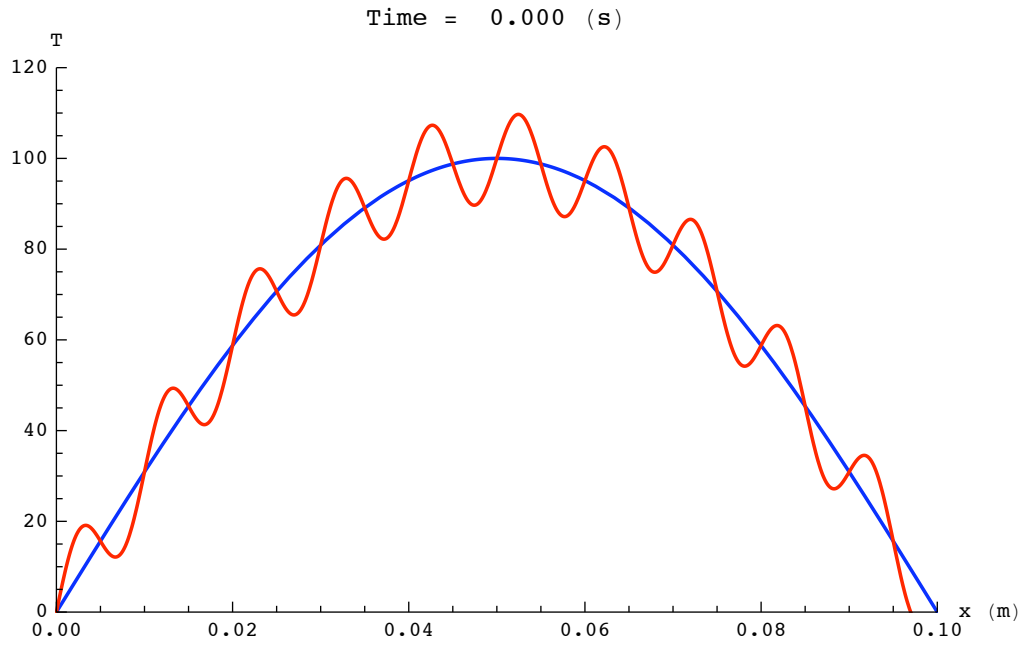
```
noisegraph[t_] := Plot[{basesol[x, t], noisesol[x, t]}, {x, 0, L},
PlotStyle->
{{RGBColor[0, 0, 1], Thickness[0.004]}, {RGBColor[1, 0, 0], Thickness[0.004]}}, ImageSi
PlotRange->{{0.0, L}, {0, 1.2*T0}}, AxesLabel-> {"x (m)", "T"},
PlotLabel -> Row[{"Time =", PaddedForm[t, {5, 3}], " (s)"}]]
```

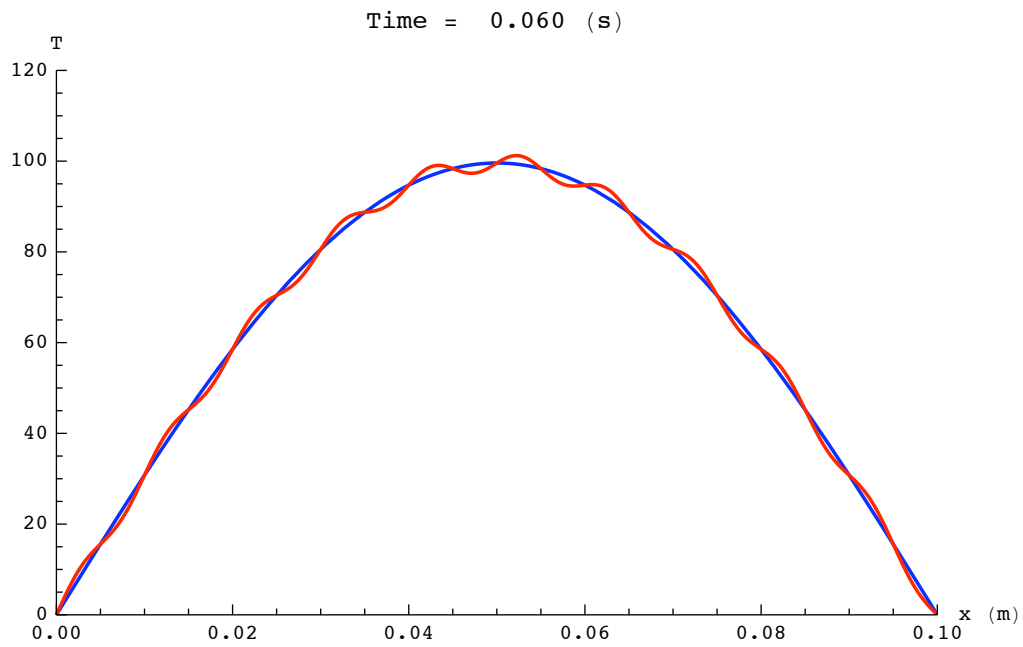
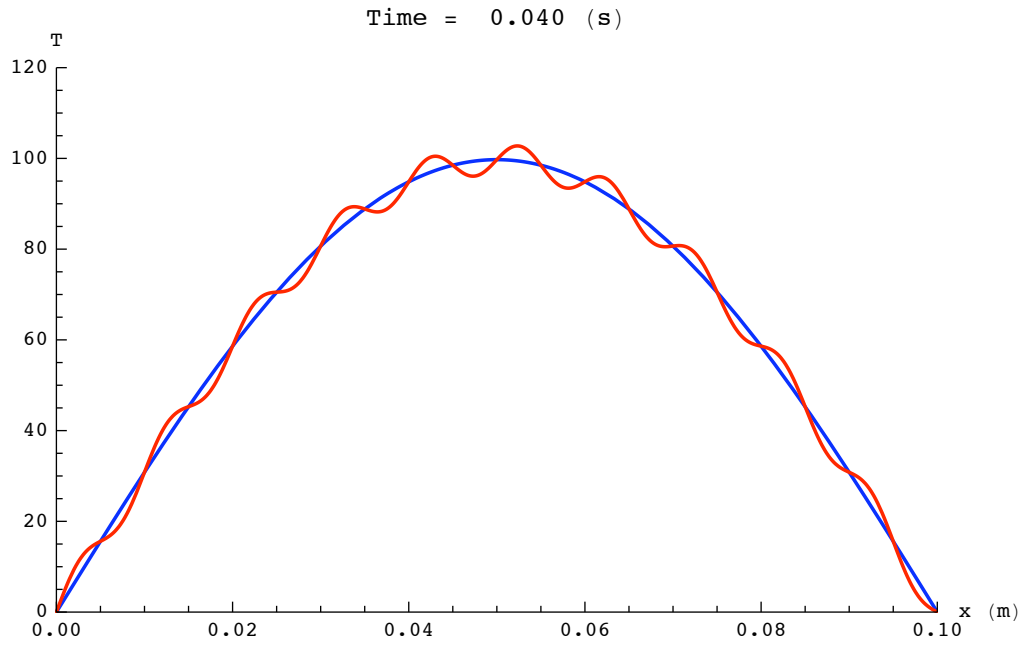
```
Do[Print[noisegraph[j]], {j, 0.0, 1.0, 0.005}]
```

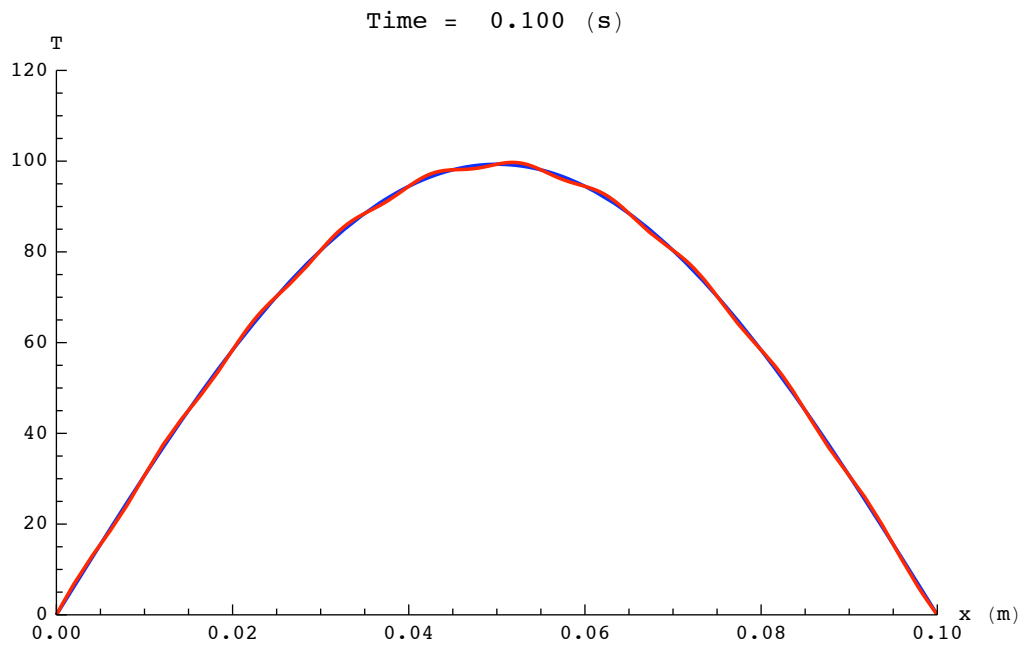
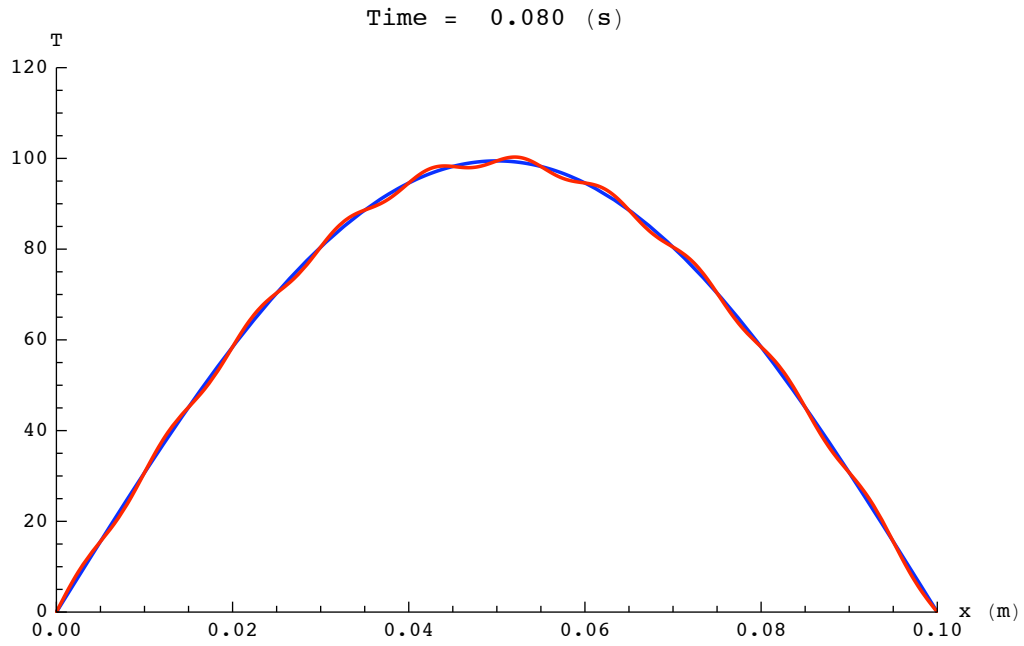


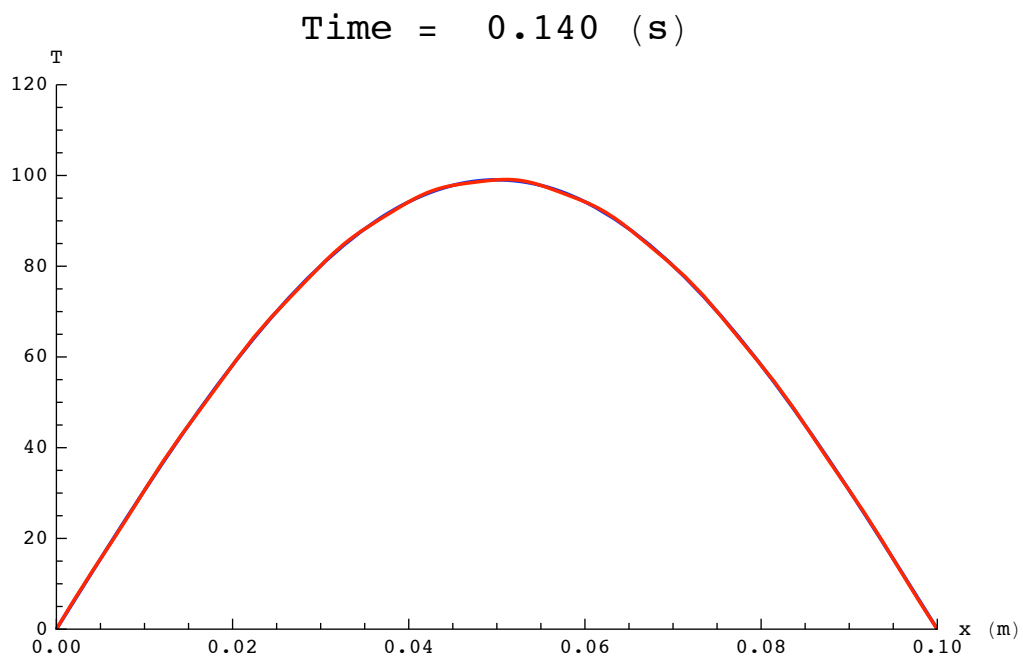
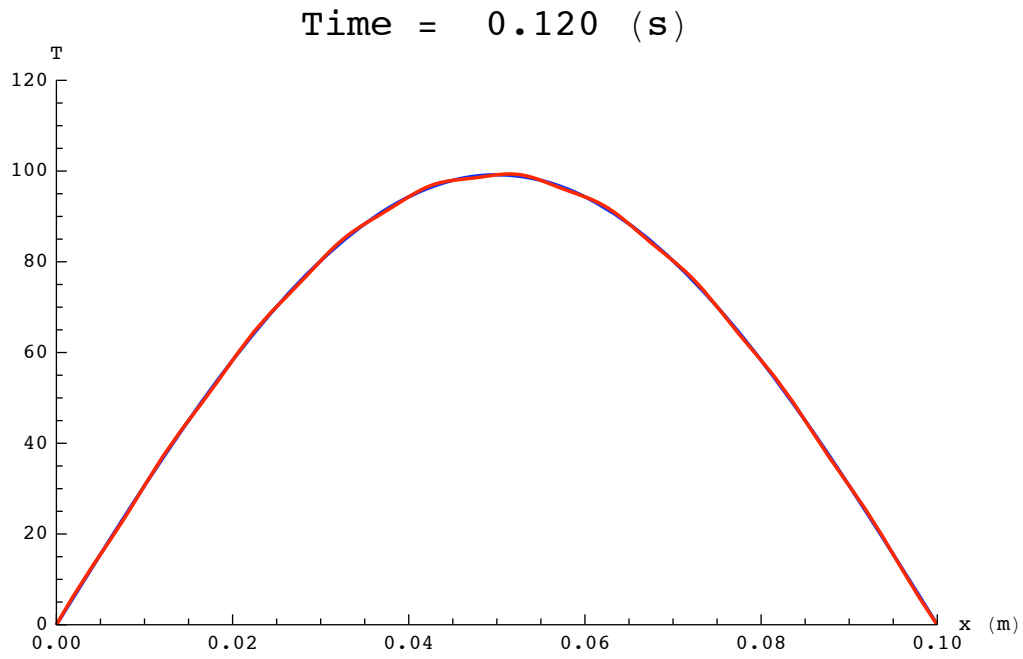
When you animate this graph sequence, you see that the small scale noise disappears very early in the time sequence, while the base solution barely moves down at all during the one second covered by the total sequence. Thus the memory of the small scale noise is lost very early in the decay process. For the printed version of this notebook, we construct a shorter sequence of 11 graphs, from  $t = 0$  to  $t = 0.2$  s at increments of 0.02 s. As we can see from the graphs, the noise is essentially gone by a time of 0.1 s.

```
Do[Print[noisegraph[j]], {j, 0.0, 0.2, 0.02}]
```

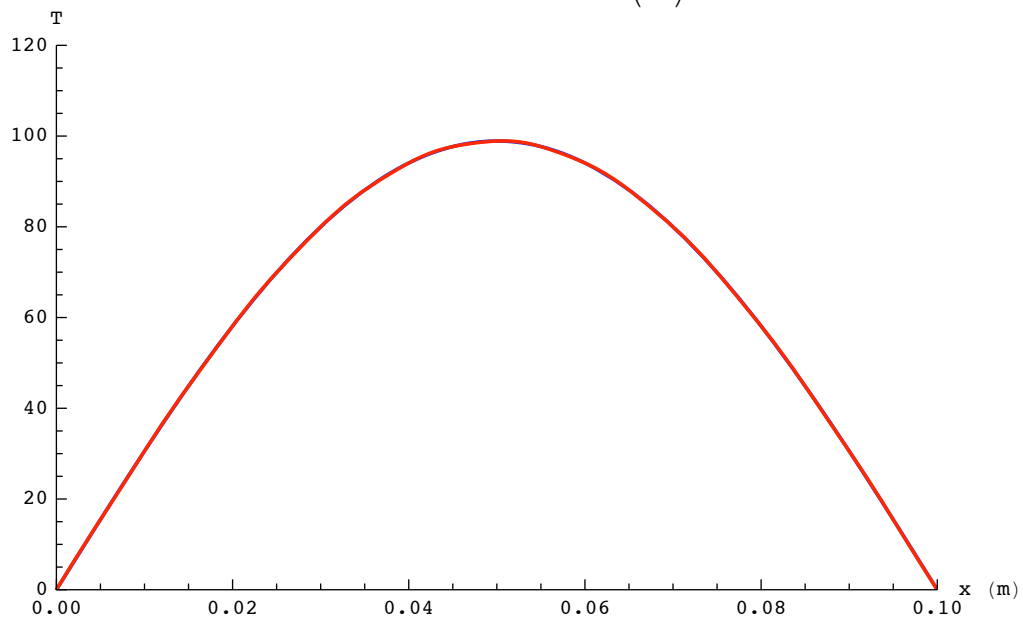








Time = 0.160 (s)



Time = 0.180 (s)

