(1) (a) We construct the Rayleigh quotient by multiplying the equation by $y$ and then integrating over $[0,1]$. 

$$\frac{\lambda}{4} y^2 \int (1 + x^2) y' \, dx + \frac{\lambda}{1 + x} x y^2 = 0$$

We integrate the first term by parts:

$$\frac{\lambda}{4} \int (1 + x^2) y' y \, dx = \left[ (1 + x^2) y y' \right] - \int (1 + x^2) y^2 \, dx$$

Then

$$\frac{\lambda}{1 + x} = x y^2 + \int (1 + x^2) y^2 \, dx$$

We integrate from 0 to 1. The last term on the right vanishes because the product $y y'$ vanishes at both 0 and 1. The result:

$$\lambda = \frac{\int_0^1 x y^2 \, dx + \int_0^1 (1 + x^2) y^2 \, dx}{\int_0^1 \frac{y^2}{1 + x} \, dx}$$

All the integrals on the right are non-negative so $\lambda \geq 0$. $\lambda = 0$ would require both numerator integrals to vanish, but $\int_0^1 x y^2 \, dx = 0 \Rightarrow y = 0$ so $\lambda > 0$.

(b) From the original equation and the general Sturm-Liouville theory, we know that the eigenfunctions are orthogonal with $\frac{1}{1 + x}$. We start with $f(x) = \sum c_k y_k(x)$. Multiply by $\frac{y_k}{1 + x}$ and integrate over $[0,1]$. We get

$$c_k = \frac{\int_0^1 f(x) \frac{y_k(x)}{1 + x} \, dx}{\int_0^1 \left[ \frac{y_k(x)}{1 + x} \right]^2 \, dx}$$

(c) The only functions satisfying the boundary conditions are those in graphs 1 and 3. From the general theory we know that the second eigenfunction has one interior zero. Hence graph 3 is the only possibility.
(2) (a) We see from the boundary conditions that the solutions will be oscillatory in \( x \). The general eigenfunctions \( u \) will then satisfy

\[
\frac{d^2 u}{dx^2} = -\lambda u, \quad 0 < x < a
\]

\[u'(0) = 0, \quad u'(a) = 0\]

We solved this problem in class. It is the problem which leads to the cosine expansion.

The eigenfunctions and eigenvalues are

\[\lambda_0 = 0, \quad u_0 = 1, \quad \lambda_n = \frac{n^2 \pi^2}{a^2}, \quad u_n = \cos \left( \frac{n \pi x}{a} \right)\]

\[n = 1, 2, 3, \ldots\]

Thus the correct expansion is (4).

(b) We substitute the expansion (4) into the equation to get

\[
\frac{d^2 u}{dy^2} + \sum_{n=1}^{\infty} \left( \frac{d^2}{dy^2} - \frac{n^2 \pi^2}{a^2} \right) u_n \cos \left( \frac{n \pi x}{a} \right) = 0.
\]

We become coefficients to get

\[
\frac{d^2 u}{dy^2} = 0, \quad \frac{d^3 u}{dy^3} - \frac{n^2 \pi^2}{a^2} u_n = 0.
\]

For \( y \) we get \( y = 0 \) gives \( A_0 = 0 \), so \( u_0 = 0 \). The homogeneous BC at \( y = 0 \) gives \( A_0 = 0 \), so \( u_0 = 0 \). The general solution for \( u_n \) is

\[u_n = A_n \sin \left( \frac{n \pi y}{a} \right) + B_n \sinh \left( \frac{n \pi y}{a} \right)\]

The BC \( \phi \bigg|_{y=0} = 0 \) gives \( A_n = 0 \). Our solution for \( \phi \) is now

\[
\phi(x, y) = B_0 y + \sum_{n=1}^{\infty} B_n \sinh \left( \frac{n \pi y}{a} \right) \cos \left( \frac{n \pi x}{a} \right).
\]
The last step is to impose the inhomogeneous BC on the top.

\[
\frac{\partial^2 \phi}{\partial y^2}(x, y) = \alpha + \delta \cos\left(\frac{5\pi y}{a}\right) = B_0 + \sum_{n=1}^{\infty} B_n \cos\left(\frac{n\pi y}{a}\right).
\]

We require coefficients to get \( B_0 = 0 \) and

\[
B_5 = \frac{\alpha \delta}{5\pi} \left(\cos\left(\frac{5\pi b}{a}\right)\right)^{-1}.
\]

So

\[
\phi(x, y) = \phi_0 + \frac{\alpha \delta}{5\pi} \frac{\sin\left(\frac{5\pi y}{a}\right)}{\left(\cos\left(\frac{5\pi b}{a}\right)\right)^{-1}} \cos\left(\frac{5\pi y}{a}\right).
\]

(3) We look for standing waves of the form

\[
\psi(x, t) = \cos(\omega t) \phi(x).
\]

We substitute this into the equation to get

\[-\omega^2 \phi = C^2 \phi''\]

or \( \phi'' + \frac{\omega^2}{C^2} \phi = 0 \)

where \( k^2 = \omega^2/C^2 \). The general solution for \( \phi \) is

\[
\phi(x) = A \cos(kx) + B \sin(kx).
\]

We impose the BC at \( x = 0; \phi''(10) = 0 = Bk \Rightarrow B = 0 \).

At \( x = l, \phi(l) = 0 = A \cos(kl) \Rightarrow k = (n - \frac{1}{2})\frac{\pi}{l} \)

\( n = 1, 2, 3, \ldots \). So \( k_n = \left(\frac{n - \frac{1}{2}}{l}\right) \pi \) and

\[
\omega_n = Ck_n = \left(\frac{n - \frac{1}{2}}{l}\right) \pi c.
\]

The fundamental \( n = 1 \) is

\[
\omega = \frac{\pi c}{2l}.
\]

The linear frequency is \( \nu = \frac{\omega}{2\pi} = \frac{c}{2l} \). Then

\[
C = 422 = 4.10.3 \times (220) = 264 \text{ m/s}.
\]