(1) We substitute \( T = F(x) G(t) \) into the equation to get
\[
(1 + \alpha x) F(x) \frac{dG}{dt} + u G(t) \frac{dF}{dx} = D G(t) \frac{d^2F}{dx^2}.
\]
We divide by \( D G \) and integrate, to get
\[
\int \frac{1}{D} \frac{dG}{dt} \, dx = -\int \frac{1}{D} u \frac{dF}{dx} \, dx + \int \frac{1}{D(1 + \alpha x)} F G(t) \, dx.
\]
The last-hand side depends only on \( t \), and the right-hand side only on \( x \), so the separation has worked. Each side is equal to the same constant, which we call \( -\lambda \). Then
\[
\text{Eqn. 1:} \quad \frac{dG}{dt} = -\lambda D G
\]
\[
\text{Eqn. 2:} \quad \frac{d^2F}{dx^2} - \frac{u}{D} \frac{dF}{dx} + \lambda (1 + \alpha x) F = 0
\]

The boundary conditions on \( F \) come from the homogeneous boundary conditions on \( T \):
\[
\frac{dF}{dx}(0) = 0, \quad F(L) = 0.
\]

(2) (a) The functions are all continuous on the base interval, so any discontinuities of the extended periodic function must show up at the points \( x = \pm 1 \).
\[
f_1(x): f_1(-1) = 0 \neq f_1(1), \text{ so extended } f_1 \text{ is discontinuous.}
\]
\[
f_2(x): f_2(-1) = 1 \neq f_2(1); f_2'(-1) = 2 = f_2'(1); f_2''(-1) = -6 \neq f_2''(1) = 0.
\]
So extended \( f_2 \), and \( f_2' \) are continuous, \( \text{Extended } f_2'' \) is discontinuous.
\[
f_3(x): f_3(-1) = 2 \neq f_3(1); f_3'(-1) = 0 \neq f_3'(1) = 4, \text{ so extended } f_3 \text{ is discontinuous, Extended } f_3' \text{ discontinuous.}
\]
Thus, \( f_2 \) has the most rapidly converging Fourier series.

The convergent will be like \( \frac{1}{\alpha^2} \).
(2) (continued) (b) \( f(x) = x - x^2 \) vanishes at \( x = 0 \) and \( x = 1 \), and it has a maximum of \( \frac{1}{4} \) at \( x = \frac{1}{2} \). The sine series represents the periodic extension of the odd extension of \( f \).

The extended function is continuous and piecewise smooth, so the Fourier sine series converges everywhere to the limit of the extended function. In particular, it converges to \( x - x^2 \) everywhere on \( [0, 2] \).

(c) The cosine series represents the periodic extension of the even extension of \( f \).

The extended function is continuous but the extended derivative is discontinuous. Therefore, the coefficients in the Fourier series will drop off like \( \frac{1}{n^2} \).

(3) Because of the homogeneous boundary conditions, there will be a non-trivial steady-state solution \( T_s(x) \). This satisfies \( \frac{d^2 T_s}{dx^2} = 0 \), \( T_s(0) = T_1 \), \( T_s(L) = T_2 \).

The equation gives \( T_s = Ax + B \). Then \( T_s(0) = T_1 \Rightarrow B = T_1 \), and \( T_s(L) = T_2 \Rightarrow A + T_1 = T_2 \Rightarrow A = (T_2 - T_1) \frac{L}{2} \), so \( T_s(x) = T_1 + \frac{T_2 - T_1}{L} x \).

We now set \( T(x,t) = T_s(x) + T'(x,t) \). The problem
(3) (continued) for $T' \text{ is then}$

$$\frac{\partial T'}{\partial t} = D \frac{\partial^2 T'}{\partial x^2}, \quad \text{at } t > 0, \quad T'(0,t) = 0, \quad T'(L,t) = 0, \quad \text{at } t > 0$$

$$T'(x,0) = T(x,0) - T_0(x) = \sum (T_3 \sin \left( \frac{3\pi x}{2L} \right) + T_5 \sin \left( \frac{5\pi x}{2L} \right))$$

$$\quad + \frac{T_3 \sin \left( \frac{3\pi x}{2L} \right)}{2} - \frac{3T_3}{4} \left( \frac{3\pi x}{2L} \right)$$

In class we used separation of variables and superposition to show that

$$T'(x,t) = \sum n \text{ e}^{-\frac{n^2 \pi^2 D t}{L^2}} \sin \left( \frac{n\pi x}{L} \right).$$

We impose the boundary conditions:

$$T'(x,0) = T_3 \sin \left( \frac{3\pi x}{2L} \right) + T_5 \sin \left( \frac{5\pi x}{2L} \right) = \sum n \sin \left( \frac{n\pi x}{L} \right)$$

$$\Rightarrow \quad (3 = T_3, \quad 5 = T_5, \quad \text{other } n = 0)$$

Then

$$T(x,t) = T'(x,t) + \frac{T_3}{L} x + \frac{T_5}{L} e^{-\frac{3\pi^2 D t}{L^2}} \sin \left( \frac{3\pi x}{2L} \right)$$

$$\quad + \frac{T_5}{L} e^{-\frac{5\pi^2 D t}{L^2}} \sin \left( \frac{5\pi x}{2L} \right)$$

(4) The diffusion time for a sphere of radius $a$ with diffusivity $D$ will be $\tau = \frac{a^2}{4D}$, where $D$ is

(which was $D^2$ for the slab) is a dimensionless constant. The cooking time will be some multiple $\tau_c$ of the diffusion time $\tau$: $\tau_c = \theta \tau_c D \approx \frac{\tau^2}{D}$

where $\tau = D/\theta$ is a dimensionless constant. If $p$ is the density and $m$ the mass, then $m = p \text{ (volume)}$ or $m = \frac{4}{3} \pi a^3 p$, so $\tau = (\frac{3}{4p})^{\frac{1}{3}} m^{\frac{1}{3}}$. Then

$$\tau_c = \frac{C}{\tau} \left( \frac{\tau^2}{D} \right)^{\frac{1}{3}} = \left[ \frac{C}{D} \left( \frac{\frac{3}{4p}}{D} \right) \right]^{\frac{1}{3}} m^{\frac{1}{3}}.$$

So the cooking time goes like $m^{\frac{1}{3}}$. 
