We start by trying \( W = y(x, y) \cos(wt) \).
Substituting into the equation, we get

\[
\nabla^4 y = \frac{w^4}{a^4} y.
\]

We look for \( y \) in the form \( y = F(x) G(y) \).

The \( x \)-operator which appears in \( \nabla^4 \) is \( a^2/\partial x^2 \),
so we look for \( F \) as an eigenfunction of

\[
\frac{a^2}{\partial x^2}.
\]

We want \( F \) and \( F'' \) to vanish
at \( x = 0 \) and \( x = a \). This problem is familiar
from other contexts, and the relevant functions
are \( F = \sin \left( \frac{m\pi x}{a} \right) \), \( m = 1, 2, 3, \ldots \).

By similar
arguments, we arrive at \( G = \sin \left( \frac{n\pi y}{b} \right) \), \( n = 1, 2, \ldots \).

Thus \( y_{mn} = \sin \left( \frac{m\pi x}{a} \right) \sin \left( \frac{n\pi y}{b} \right) \),
and we have

\[
\nabla^4 y_{mn} = \nabla^2 (\nabla^2 y_{mn}) = \nabla^2 \left( -\frac{m^2}{a^2} \right) \sin \left( \frac{m\pi x}{a} \right) \sin \left( \frac{n\pi y}{b} \right).
\]

\[
= \left( \frac{m^2}{a^2} + \frac{n^2}{b^2} \right)^2 y_{mn}.
\]

The equation for \( y \) now becomes

\[
\omega_{mn}^2 = \sigma \left( \frac{m^2}{a^2} + \frac{n^2}{b^2} \right)
\]

so \( \omega_{mn} = \sqrt{\sigma} \left( \frac{m^2}{a^2} + \frac{n^2}{b^2} \right) \).

This is the angular frequency of mode \( mn \).

It is worth noting that with other boundary
conditions (clamped edge, for example), the analysis
is much more difficult.
(b) For \( b = 2a \), \( \omega_{mn} = \sqrt{\frac{5}{b^2}} \left( \frac{4m^2 + n^2}{b^2} \right) \).

The factor which varies from mode-to-mode is \( (4m^2 + n^2) \). Here is a short table of values:

<table>
<thead>
<tr>
<th>( m )</th>
<th>1</th>
<th>2</th>
<th>3</th>
<th>4</th>
</tr>
</thead>
<tbody>
<tr>
<td>( n )</td>
<td>5</td>
<td>17</td>
<td>37</td>
<td>65</td>
</tr>
<tr>
<td>2</td>
<td>8</td>
<td>20</td>
<td>40</td>
<td>68</td>
</tr>
<tr>
<td>3</td>
<td>13</td>
<td>25</td>
<td>45</td>
<td>73</td>
</tr>
<tr>
<td>4</td>
<td>20</td>
<td>32</td>
<td>52</td>
<td>80</td>
</tr>
</tbody>
</table>

We see that the four modes with the lowest frequencies are \((m=1, n=1)\), \((m=1, n=2)\), \((m=1, n=3)\) and \((m=2, n=1)\). The modes, frequencies and node lines are shown below.

\[ \frac{\omega_{11}}{b^2} = \frac{5\sqrt{2} \pi}{b^2} \]

There are no node lines other than the plate boundaries.

\[ \frac{\omega_{12}}{b^2} = \frac{8\sqrt{2} \pi}{b^2} \]

The \( y \)-function vanishes at \( y = \frac{b}{2} \).
(1) (b) \( m=1, n=3 \) \( \omega_{13} = \frac{13 \sqrt{9 \pi^2}}{b^2} \)

\[ W_{13} = \cos \left( \omega_{13} t \right) \sin \left( \frac{\pi x}{a} \right) \sin \left( \frac{3 \pi y}{b} \right) \]

The \( y \)-function vanishes at \( \frac{b}{3} \) and \( \frac{2b}{3} \).

\[ m = 2, n = 1 \] \( \omega_{11} = \frac{17 \sqrt{8 \pi^2}}{b^2}, \quad W_1 = \cos \left( \omega_{11} t \right) \sin \left( \frac{2 \pi y}{b} \right) \sin \left( \frac{\pi x}{a} \right) \]

The \( x \)-function vanishes at \( x = \frac{a}{2} \).

(C) We use consistent English units of pounds for force, slugs for mass, foot for length and seconds for time. Then

\[ E = 30 \times 10^6 \text{ lb/} \text{in}^2 = 30 \times 10^6 \times 144 \text{ lb/ft}^2 = 4.32 \times 10^9 \text{ lb/ft}^2 \]

\[ \rho = 490 \text{ lb/} \text{ft}^3 = \frac{490}{32.2} \frac{\text{slugs}}{\text{ft}^3} = 15.27 \text{ slugs/ft}^3 \]

Then

\[ \sigma = \frac{E \frac{b^2}{12 \rho (1-\nu)^2}}{12 (15.27)(1-0.3)^2} = 2599.75 \frac{\text{lb}}{\text{in}^2} \]

The linear frequencies in Hz

\[ \frac{1}{\omega_m} = \frac{\text{lb/in}}{2\pi} \frac{\sqrt{\pi^2 - \pi^2}}{2b^3} \left( \sin^2 \frac{\pi y}{b} \right) = 80.091 \left( \sin^2 \frac{\pi y}{b} \right) \text{ s}^{-1}. \]
(1) (continued) The four frequencies are
$$\nu_1 = 400.4 \text{ Hz}, \quad \nu_2 = 690.7 \text{ Hz},$$
$$\nu_3 = 1040 \text{ Hz}, \quad \text{and} \quad \nu_4 = 1362 \text{ Hz}.$$ 

(2) To solve this problem, we need to find the standing modes and their frequencies. The motion functions will be eigenfunctions of $\Delta^2 \phi = 0$ or $\Delta^2 \phi = 0$, and they are required by the boundary conditions to vanish at the boundaries. This leads to our familiar sine eigenfunctions, so we expect modes of the form
$$\phi_m(x, y, t) = \cos(\omega_n x) \sin\left(\frac{m\pi x}{L}\right) \sin\left(\frac{n\pi y}{B}\right).$$
$m, n = 1, 2, \ldots$

(This form may be derived by separation of variables.) Substituting this into the equation for $U$ gives
$$-\omega_n^2 U = C^2 \left(\frac{m^2}{L^2} + \frac{n^2}{B^2}\right).$$

The first lower frequency is given by
$$\omega_n = \frac{C}{L} \left(\frac{m^2}{L^2} + \frac{n^2}{B^2}\right).$$

Then modes with frequencies less than or equal to $\omega_n$ satisfy
$$\frac{C^2}{L^2} \left(\frac{m^2}{L^2} + \frac{n^2}{B^2}\right) \leq \nu_n^2,$$

or
$$\left(\frac{m}{2L/L} \right)^2 + \left(\frac{n}{2B/B}\right)^2 \leq 1.$$
(2) (continued) \( IS = \frac{1}{4} \pi \left( \frac{2\pi v}{C} \right) \left( \frac{2\pi h}{C} \right) = \frac{\pi^2 v_0^2 \omega}{C^2} \)

Because each mode occupies unit area in the \(2\pi\) space, the number of modes with frequency \( \leq v_0 \) is
\[ N(v_0) = \frac{\pi^2 v_0^2 \omega}{C^2} \]

Then the number of modes with frequency \( \leq v \) in the range \( v_0 \leq v < v_1 \) is
\[ N(v_1) - N(v_0) = \frac{\pi^2 \omega}{C^2} \left( v_1^2 - v_0^2 \right) \]
\[ = \frac{\pi^2 (3)(3)}{(10^3/\omega)} \left( (4000)^2 - (2000)^2 \right) \]
\[ = 22,620 \text{ modes.} \]

(3) (a) \( \tilde{f}(k) = \int_{-\infty}^{\infty} e^{-ikx} e^{-ix} \cos(x) \, dx \)
\[ = \int_{-\infty}^{\infty} \cos(kx) e^{-ix} \cos(x) \, dx \]
\[ = 2 \int_{0}^{\infty} \cos(kx) e^{-x} \cos x \, dx \]

It is straightforward to do this with basic calculus techniques, but it is even easier to do it with Mathematica. We do it with Mathematica here. See the Mathematica notebook. The result is
\[ \tilde{f}(k) = \frac{2 (2 + k^2)}{4 + k^2} \]

(b) \( \tilde{f}(0) = 1 \cdot \int_{-\infty}^{\infty} f(x) \, dx = \int_{-\infty}^{\infty} e^{-ix} \cos x \, dx \)
\[ = 2 \int_{0}^{\infty} e^{-x} \cos x \, dx = 1 \]
(See Mathematica notebook)
The integral is the left is
\[
\int_{-\infty}^{\infty} e^{-2ix} \cos^2(x) \, dx = 2 \int_{0}^{\infty} e^{-2x} \cos^2 x \, dx = \frac{3}{4} \quad \text{(see Mathematica notebook.)}
\]

The integral on the right is
\[
\frac{1}{2\pi} \int_{-\infty}^{\infty} \frac{4(2+4k^2)}{(2+4k^2)^2} \, dk = \frac{2}{2\pi} \int_{0}^{\infty} \frac{(2+4k^2)}{(2+4k^2)^2} \, dk = \frac{3}{4} \quad \text{(see Mathematica notebook.)}
\]

(4) (b) The Fourier transform of the equation is
\[
\mathcal{F}\{y''\} - 2 \mathcal{F}\{y'\} = \mathcal{F}\{e^{-x} \cos x\}
\]
\[-k^2 \tilde{y} - 2 \tilde{y} = \frac{2}{2+4k^2} \]

So
\[
\tilde{y} = -\frac{1}{2+4k^2} \cdot \frac{2}{2+4k^2} = -\frac{2}{2+4k^2}
\]

(b) The inversion integral gives
\[
y(x) = \frac{1}{2\pi} \int_{-\infty}^{\infty} e^{ixk} \tilde{y}(k) \, dk = -\frac{1}{2\pi} \int_{-\infty}^{\infty} e^{ixk} \cdot \frac{2}{2+4k^2} \, dk
\]
\[
= -\frac{1}{2\pi} \int_{-\infty}^{\infty} \frac{\cos 2kx}{2+4k^2} \, dk = -\frac{2}{2\pi} \int_{0}^{\infty} \frac{\cos 2kx}{2+4k^2} \, dk
\]
\[
= -\frac{1}{4} \left[ \cos(x) + \sinh(x) \right] \left[ \cos(x) - \sinh(x) \right] \quad \text{(see Mathematica notebook.)}
\]
We use Mathematica to check these solutions. We have

\[ y(x) = y(x) = -\frac{1}{4} \left[ \cos(x) + \sin(x) \right] \left[ \cos(\bar{x}) - \sin(\bar{x}) \right] \]

for \( x > 0 \)

and \( y(x) = y(x) = -\frac{1}{4} \left[ \cos(x) - \sin(x) \right] \left[ \cos(\bar{x}) + \sin(\bar{x}) \right] \)

for \( x < 0 \).

These simplify to \( y_1(x) = -\frac{e^{-x}}{4} \left[ \cos(x) + \sin(x) \right] \)

and \( y_2(x) = -\frac{e^{x}}{4} \left[ \cos(x) - \sin(x) \right] \).

We use Mathematica to check that \( y_1(x) \) is a solution for \( x > 0 \), to check that \( y_2(x) \) is a solution for \( x < 0 \), and to check that the solution is sufficiently smooth at \( x = 0 \) (continuity of \( y, y', \) and \( y'' \) at \( x = 0 \)). We also graph the solution.

**Challenge Problem**

(a) We substitute \( y_n = F_n(t) \sin(n \omega t) \) in the equation. We get

\[ F_n'' + \epsilon F_n' + (\omega_n^2) F_n = 0 \]

where \( \omega_n = \frac{n \omega}{2} \) is the natural frequency of the \( n \)th undamped mode. We try \( F_n = e^{rt} \) to get

\[ r^2 + \epsilon r + \omega_n^2 = 0 \]

which has the roots

\[ r_1 = -\frac{1}{2} \epsilon + \sqrt{\frac{\epsilon^2}{4} - \omega_n^2}, \quad r_2 = -\frac{1}{2} \epsilon - \sqrt{\frac{\epsilon^2}{4} - \omega_n^2}. \]
The solution is then \( F(t) = A_0 e^{\frac{\pi}{2} t} + B_0 e^{-\frac{\pi}{2} t} \).

(b) For \( \varepsilon > \frac{\pi c}{l} \), we have \( \frac{\varepsilon^2}{4} > \frac{\pi^2 c^2}{l^2} = \omega_1^2 > \omega_2^2 \),

so \( \frac{\varepsilon^2}{4} > \omega_2^2 \) for \( \varepsilon > \frac{\pi c}{l} \), and the square root in the formulas for \( \kappa_1 \) and \( \kappa_2 \) are imaginary. Then,

\[ \kappa_1 = -\frac{1}{2} \varepsilon + i\beta_0, \quad \kappa_2 = -\frac{1}{2} \varepsilon - i\beta_0 \]

where \( \beta_0 = \sqrt{\omega_2^2 - \frac{\varepsilon^2}{4}} \). A convenient form of the solution for \( F_0 \) is then

\[ F_0(t) = e^{-\frac{\varepsilon t}{2}} \left( A_0 \cos \beta_0 t + B_0 \sin \beta_0 t \right) \]

where \( A_0, B_0 \) are arbitrary and are needed to satisfy initial conditions (which we are not concerned with in this problem). Thus (b) of the modes are damped oscillations in time.

If \( \varepsilon > \frac{\pi c}{l} \), then \( \frac{\varepsilon^2}{4} > \frac{\pi^2 c^2}{l^2} = \omega_1^2 \).

So for at least the first mode, the solutions are pure exponentials:

\[ \kappa_1 = -\frac{1}{2} \varepsilon + \sqrt{\frac{\varepsilon^2}{4} - \omega_1^2} < 0 \]

and \( \kappa_2 = -\frac{1}{2} \varepsilon - \sqrt{\frac{\varepsilon^2}{4} - \omega_1^2} < 0 \).

Thus both terms in the solutions are damped exponentials. Let \( N = \text{largest integer} < \frac{\varepsilon l}{2 \pi c} \).

Then for \( n \leq N \), the mode is a damped oscillating mode is a damped oscillation.
(c) For \( E \ll 2 \pi C \), \( \frac{E^2}{L} \ll \frac{\pi^2 C^2}{L^2} = \omega_0^2 \), \( \omega \ll \omega_0 \).

So \( \sqrt{\frac{E^2}{\omega_0^2} - \omega^2} = i \sqrt{\omega_0^2 - \frac{\omega^2}{\omega_0^2}} \leq i \sqrt{\omega_0^2} = \pm i \omega_0 \).

Then \( r = -\frac{1}{2} \epsilon \pm i \omega_0 \) and the solution for \( F_n(t) \) is

\[
F_n(t) = e^{-\frac{t}{2\epsilon}} \left[ A_n \cos(\omega_0 t) + B_n \sin(\omega_0 t) \right].
\]

Thus, the damping is first order in the sense that the frequency correction is second order in \( \epsilon \), and is ignored in this approximation.

(d) We assume that we can use the approximation of part (c). We will justify this at the end of our calculation. Then

\[
\frac{1}{2} = e^{-\frac{t}{2\epsilon}}
\]

where \( \omega = 8 \text{ seconds} \). Then

\[
L_n \omega = \frac{1}{2} \epsilon \omega
\]

\[
\epsilon = \frac{2L_n \omega^2}{\omega} = 0.173.
\]

For this string \( L_n = 330 \text{ Hz} \), so \( \omega_1 = 2\pi L_n = 2073 \text{ s}^{-1} \).

Then \( \epsilon \ll \frac{2\pi C}{L} \Rightarrow \frac{E}{2\pi C} \ll \frac{1}{L} \Rightarrow \frac{E}{2\omega} < 1 \).

And \( \frac{E}{2\omega_1} = \frac{0.173}{2 (2073)} = 4.17 \times 10^{-5} \),

so the approximation is justified.
Assignment #8 Solutions
Problems 3 and 4

Problem 3

(a)

\[
\text{In}[1]:= \text{f[k_] = Integrate}[2 \text{Cos}[k x] \cos[x] \text{Exp}[-x], \{x, 0, \infty\}, \text{Assumptions} \rightarrow k \in \text{Reals}]
\]

\[
\text{Out}[1]= \frac{2 (2 + k^2)}{4 + k^4}
\]

(b)

\[
\text{In}[2]:= f[0]
\]

\[
\text{Out}[2]= 1
\]

\[
\text{In}[3]:= \text{Integrate}[2 \text{Exp}[-x] \cos[x], \{x, 0, \infty\}]
\]

\[
\text{Out}[3]= 1
\]

(c)

\[
\text{In}[4]:= \text{Integrate}[2 \text{Exp}[-2 x] (\cos[x])^2, \{x, 0, \infty\}]
\]

\[
\text{Out}[4]= \frac{3}{4}
\]

\[
\text{In}[5]:= (4/\pi) \text{Integrate} \left[ \frac{(2 + k^2)^2}{(4 + k^4)^2}, \{k, 0, \infty\} \right]
\]

\[
\text{Out}[5]= \frac{3}{4}
\]
Problem 4

(b)

\[
\text{In[6]} = - \left( \frac{2}{\pi} \right) \text{Integrate} \left[ \frac{\cos[kx]}{4 + k^4}, \{k, 0, \infty\}, \text{Assumptions} \rightarrow x \in \text{Reals} \right]
\]

\[
\text{Out[6]} = - \frac{1}{4} \left( \cos[x] + \sin[\text{Abs}[x]] \right) \left( \cosh[x] - \sinh[\text{Abs}[x]] \right)
\]

We now check the solution. The solution for \( x \) positive is

\[
\text{In[7]} = y_R'[-x] = - \frac{\exp[-x]}{4} (\cos[x] + \sin[x]);
\]

The solution for \( x \) negative is

\[
\text{In[8]} = y_L'[-x] = - \frac{\exp[x]}{4} (\cos[x] - \sin[x]);
\]

We start by checking the solution for \( x > 0 \).

\[
\text{In[9]} = \text{D}[y_R[x], \{x, 2\}] - 2 y_R[x] - \exp[-x] \cos[x]
\]

\[
\text{Out[9]} = - e^{-x} \cos[x] - \frac{1}{4} e^{-x} (-\cos[x] - \sin[x]) + \frac{1}{2} e^{-x} (\cos[x] - \sin[x]) + \frac{1}{4} e^{-x} (\cos[x] + \sin[x])
\]

\[
\text{In[10]} = \text{Simplify}[\%]
\]

\[
\text{Out[10]} = 0
\]

That checks. Now we check the solution for negative \( x \).

\[
\text{In[11]} = \text{D}[y_L[x], \{x, 2\}] - 2 y_L[x] - \exp[x] \cos[x]
\]

\[
\text{Out[11]} = - e^x \cos[x] - \frac{1}{2} e^x (-\cos[x] - \sin[x]) + \frac{1}{4} e^x (\cos[x] - \sin[x]) - \frac{1}{4} e^x (-\cos[x] + \sin[x])
\]

\[
\text{In[12]} = \text{Simplify}[\%]
\]

\[
\text{Out[12]} = 0
\]

That also checks. Now we check the smoothness of \( y \) at zero. We expect \( y, y' \) and \( y'' \) to be continuous at \( x = 0 \). We do not expect \( y''' \) to be continuous at \( 0 \). If you differentiate the equation with respect to \( x \), you see that one of the terms in the resulting expression for \( y''' \) is the derivative of the right hand side, a function which is discontinuous at \( x = 0 \).

\[
\text{In[13]} = y_R[0] - y_L[0]
\]

\[
\text{Out[13]} = 0
\]

\[
\text{In[14]} = y_R'[0] - y_L'[0]
\]

\[
\text{Out[14]} = 0
\]

\[
\text{In[15]} = y_R''[0] - y_L''[0]
\]

\[
\text{Out[15]} = 0
\]
Everything is as expected. Now we plot the solution.

The solution is an even function, as we see either from the formulas or the graph. It is easy to show that the minimum of \(-1/4\) occurs at \(x = 0\), and that the two maxima on the graph are \(e^{-\pi/4}\) at \(x = \pm\pi\). There are an infinity of other maxima and minima for \(x\) outside the range shown on the graph, but the function is decaying to zero and the values are very small.