(i) (a) We construct the Rayleigh quotient. Multiply the equation by $F$:

$$F(x^2F')' = -2x^3F$$
$$F(x^2F') = -x^5(F')^2 = -\lambda x^3F$$

Integrate over $[1, 3]$ and use the fact that $F=0$ at $x=1$ and $x=3$. Then

$$\lambda = \frac{\int_1^3 x^5 F'^2 \, dx}{\int_1^3 x^3 F^2 \, dx} \geq 0.$$ 

This gives $\lambda = 0$ if and only if $F'=0 \Rightarrow F=\text{constant}$, but the boundary conditions tell us the constant is zero. So $\lambda = 0$ gives only the trivial solution.

(b) We write the equation as

$$x^5 F'' + 5x^4 F' = -\lambda x^3 F.$$ 

Each term has the same dimensions in $x$ - obviously $x^3$, so there will be solutions of the form $F=x^r$. We substitute this into the equation to get

$$r(r-1)x^{r+3} + 5r x^{r+3} = -\lambda x^{r+3}.$$ 

So

$$r(r+3) = -\lambda,$$

$$(r+2)^2 = r^2 - \lambda$$

$$r = -2 \pm \sqrt{\lambda - 4}.$$ 

Let $\alpha = \sqrt{\lambda - 4}$. Then our solutions are

$$x^{-2+\alpha} = x^{-2} \left[ \cos(\alpha \ln x) + i \sin(\alpha \ln x) \right]$$

and

$$x^{-2-i\alpha} = x^{-2} \left[ \cos(\alpha \ln x) - i \sin(\alpha \ln x) \right].$$

A more useful real-valued solution basis is $x^{-2}\cos(\alpha \ln x)$ and $x^{-2}\sin(\alpha \ln x)$.

The general solution is $F = A x^{-2}(\cos(\alpha \ln x)) + B x^{-2} \sin(\alpha \ln x)$.

We impose the BC's. $F(0) = 0 \Rightarrow A = 0$

$F(3) = 0 \Rightarrow (8/9) \sin(\alpha \ln 3) = 0 \Rightarrow \alpha \ln 3 = \pi n,$

$n = 1, 3, 5, \ldots$.
According to the general theory, two eigenfunctions associated with different eigenvalues are orthogonal with respect to the weight function, which is $x^3$ in this case.

$$\int_{-1}^{1} F_n(x) F_m(x) x^3 dx = \int_{-1}^{1} \sin \left( \frac{\pi n x}{a_3} \right) \sin \left( \frac{\pi m x}{a_3} \right) \frac{dx}{x^3}$$

Let $y = \frac{bx}{a_3}$. Then $dx = b_3 dy$ and the integral is

$$b_3 \int_{-1}^{1} \sin \left( \pi n y \right) \sin \left( \pi m y \right) dy.$$  

This is a familiar integral from our work with Fourier series, and we know it is zero for $m \neq n$.

(d) See Mathematical Notebook.

(e) $f(x) = x^2 = \sum \alpha_n F_n(x)$. We use orthogonality to calculate the coefficients. Multiply the equation by $x^3 F_k(x)$ and integrate over $[-1,1]$. You get

$$C_k = \frac{\int_{-1}^{1} x \cdot F_k(x) dx}{\int_{-1}^{1} x^3 [F_k(x)]^2 dx}$$

We may evaluate the denominator by our orthogonality calculation above:

$$\text{denominator} = b_3 \int_{-1}^{1} \sin^2 (\pi n y) dy = \frac{1}{2} b_3.$$
The numerator is

\[ \int_{x}^{3} F(x) \, dx = \int_{0}^{3} \sin \left( k \pi \frac{b \omega x}{a} \right) \frac{dx}{x} \]

\[ = \int_{0}^{3} \frac{b \omega}{k \pi} \sin (k \pi y) \, dy \]

\[ = \frac{b \omega}{k \pi} \left[ -\frac{1}{k \pi} \right] = \frac{b \omega}{k \pi}, \quad k \text{ odd} \]

\[ = 0, \quad k \text{ even} \]

So \( C_k = \frac{a}{k \pi} \) for \( k \text{ odd} \)

\[ = 0 \quad \text{for} \quad k \text{ even} \]

Then

\[ F(x) = \frac{1}{x^2} = \sum_{k=1}^{\infty} \frac{1}{k \pi} \sin (k \pi \frac{b \omega x}{a}) \]

(This is the Fourier sine series for the square wave, in disguise.)

See Mathematica notebook for plot of partial sum.

(f) Suppose \( \lambda < 4 \). Then \( F(x) = x^r \) where \( r = 1 \pm \sqrt{4 - \lambda} \)

So the general solution is \( F(x) = x^r \left\{ A x^{\frac{r}{2}} + B x^{-\frac{r}{2}} \right\} \)

\( F(1) = 0 \Rightarrow A = -B \Rightarrow F(x) = A x^r \left\{ x^{\frac{r}{2}} - x^{-\frac{r}{2}} \right\} \)

Then \( F(3) = 0 \Rightarrow 3^{\frac{r}{2}} - 3^{-\frac{r}{2}} = 0 \) or \( 3^r = 1 \Rightarrow r = 0 \).

So \( \lambda < 4 \) is not possible. For \( \lambda = 4 \), the characteristic equation \((r + 2)^2 = 4 - \lambda \) has a repeated root \( r = 2 \). Then the solutions are \( x^2 \) and \( b x \cdot x^{-2} \). The general solution is \( F(x) = A x^2 + B b x \cdot x^{-2} \). We have \( F(1) = A = 0 \), and \( F(3) = 0 \Rightarrow B b 3^{-2} = 0 \Rightarrow B = 0 \), so there is no solution for \( \lambda = 4 \).

\[ \lambda > 5 \]
(2) (a) We try \( \Phi = F(x) G(t) \). We substitute into the equation and divide by \( FG x^3 \) to get:

\[
\frac{1}{G} \frac{dG}{dt} = -\frac{1}{x^3} \frac{d}{dx} \left( x^5 \frac{dF}{dx} \right).
\]

The separation has worked, so both sides are equal to the same constant which we call \(-\lambda\).

\[
\frac{d}{dx} \left[ x^5 \frac{dF}{dx} \right] = -\lambda x^3 F, \quad x \neq 0.
\]

We imposed on \( F \) the original homogeneous boundary conditions on \( \Phi \). This is the problem that we just solved:

\[
\lambda_n = 4 + \left( \frac{n\pi}{203} \right)^2, \quad F_n(x) = \frac{1}{x} \sin \left[ \frac{n\pi - 40x}{203} \right].
\]

\( n = 1, 2, 3, \ldots \).

\( t \)-equation:

\[
\frac{dG_n}{dt} = -\lambda_n G_n \quad \text{so} \quad G_n(0) = C.
\]

(Apart from a multiplicative constant.)

We superpose our solutions:

\[
\Phi(x,t) = \sum_{n=1}^{\infty} C_n e^{-\lambda_n t} F_n(x).
\]

Finally we impose the initial condition:

\[
\Phi(x,0) = \frac{1}{x^2} = \sum_{n=1}^{\infty} C_n F_n(x).
\]

We already obtained this expansion in problem 1, and

\[
C_n = \begin{cases} 
\frac{4}{n\pi} & \text{for } n \text{ odd} \\
0 & \text{for } n \text{ even}
\end{cases}
\]
\( \Phi(x,t) = \sum_{n=1}^{\infty} \frac{4}{n \pi^2} e^{-\lambda_n t} \sin \left[ \frac{n \pi x}{b_3} \right] 

\begin{align*}
&= \frac{4}{\pi^2 t} e^{-\lambda_1 t} \sin \left[ \frac{\pi x}{b_3} \right] \\
&\quad + e^{-\lambda_2 t} \sin \left[ \frac{3 \pi x}{b_3} \right] \\
&\quad + \frac{1}{5} e^{-\lambda_5 t} \sin \left[ \frac{5 \pi x}{b_3} \right] + \ldots
\end{align*}

We can approximate this by the first term only whenever \( t \) is large enough to make
\[ \frac{e^{-\lambda_1 t}}{e^{-\lambda_2 t}} \ll 1. \]

In using this argument we are using the fact that the \( \lambda_n \)'s increase rapidly and monotonically with \( n \).

We have
\[ \frac{e^{-\lambda_2 t}}{e^{-\lambda_3 t}} = e^{-\frac{\lambda_2^2}{\frac{(b_3)^2}{\pi^2}} t}. \]

If \( t \geq \left( \frac{b_3}{\pi} \right)^2 \), then the exponential ratio is smaller than \( e^{-B} \).

Then
\[ \Phi(x,t) \approx \frac{4}{\pi^2 x^2} \sin \left[ \frac{\pi x}{b_3} \right] e^{-\lambda_1 t} e^{-\left( \frac{b_3}{\pi} \right)^2 t}. \]
CHALLENGE PROBLEM

(a) Any $y$ in $S$ is twice continuously differentiable and satisfies the boundary condition satisfied by the $y_n$'s. Hence we have

$$y(x) = \sum_{n=1}^{\infty} c_n y_n(x) \text{ on } [a, b],$$

with

$$c_n = \frac{\int_a^b P(x) y(x) y_n(x) \, dx}{\int_a^b P(x) [y_n(x)]^2 \, dx}.$$

We also know that

$$L y = \sum_{n=1}^{\infty} \lambda_n c_n y_n = \sum_{n=1}^{\infty} \lambda_n c_n y_n(x).$$

We use this to calculate the numerator of the functional:

$$-\int_a^b y L y \, dx = \int_a^b y \sum_{n=1}^{\infty} \lambda_n c_n P y_n \, dx = \sum_{n=1}^{\infty} \lambda_n c_n \int_a^b P y y_n \, dx

= \sum_{n=1}^{\infty} \lambda_n c_n c_n N_n = \sum_{n=1}^{\infty} \lambda_n c_n^2 N_n,$$

where $N_n = \int_a^b P y_n^2 \, dx$. The denominator is

$$\int_a^b P y^2 \, dx = \int_a^b P y \left(\sum_{n=1}^{\infty} c_n y_n\right) \, dx = \sum_{n=1}^{\infty} c_n \int_a^b P y y_n \, dx

= \sum_{n=1}^{\infty} c_n c_n N_n = \sum_{n=1}^{\infty} c_n^2 N_n.$$

So $F[y] = \frac{\sum_{n=1}^{\infty} \lambda_n c_n^2 N_n}{\sum_{n=1}^{\infty} c_n^2 N_n}$. 


CHALLENGE PROBLEM (continued)

(b) We have $0 < \lambda_1 < \lambda_2 < \lambda_3 \ldots \quad \text{so}$

$$\sum_{n=1}^{\infty} \lambda_n \ell_n^2 N_n \geq \sum_{n=1}^{\infty} \lambda_1 \ell_n^2 N_n = \lambda_1 \sum_{n=1}^{\infty} \ell_n^2 N_n,$$

where we have used $\ell_n^2 \geq 0$, $N_n > 0$. Then

$$F[y] = \frac{\sum_{n=1}^{\infty} \lambda_n \ell_n^2 N_n}{\sum_{n=1}^{\infty} \ell_n^2 N_n} \geq \frac{\lambda_1 \sum_{n=1}^{\infty} \ell_n^2 N_n}{\sum_{n=1}^{\infty} \ell_n^2 N_n} = \lambda_1.$$

Thus any function from $S$ that we substitute into $F$ produces a number $\geq \lambda_1$.

(c) For this Sturm-Liouville system

$$F[y] = \frac{\int_0^\infty y''^2 \, dx}{\int_0^\infty y'^2 \, dx}.$$ 

We try $y = x(1-x)$. Then $y' = 1-2x$ and

$$F[y] = \frac{\int_0^\infty (1-2x)^2 \, dx}{\int_0^\infty x^2(1-x)^2 \, dx} = \frac{1/3}{1/10} = 10.$$

This exceeds $\lambda = 9.8696$ only by about 1.3% - a surprisingly good result for such a simple approximation.

(d) Now we try $y = x(1-x) + ax^2(1-x)^2$.

We use Mathematica to do the integrals. We get

$$F[y] = \frac{\frac{1}{3} + \frac{2a}{15} + \frac{2a^2}{105}}{ \frac{1}{30} + \frac{a}{70} + \frac{a^2}{630}}$$

As shown in the Mathematica notebook, the minimum occurs for $a = 1.133$, giving $F = 9.86975$ - giving 4 digits correctly and a small error in the fifth digit.
(e) The functional is \( \lambda = \text{num} / \text{den} \), where

\[
\text{num} = \int_a^b (ry'' + gy'^2) \, dx, \quad \text{den} = \int_a^b ry^2 \, dx
\]

Then

\[
\delta \lambda = \frac{\delta \text{num}}{\delta \text{den}} - \frac{\text{num}}{\text{den}} \delta \text{den} = 0
\]

so

\[
\delta \text{num} - \frac{\text{num}}{\text{den}} \delta \text{den} = 0
\]

or

\[
\delta \text{num} - \lambda \delta \text{den} = 0.
\]

We have \( \delta \text{num} = \int_a^b (2ry' sy' + 2gsy) \, dx \)

\[
= \int_a^b \left( (2ry' sy')' - 2(ry' sy' + 2gsy) \right) \, dx
\]

The first term integrates to zero because \( sy = 0 \) at the endpoints, so

\[
\delta \text{num} = -\int_a^b \left[(ry')' - gy\right] sy \, dx
\]

\[
\delta \text{den} = \int_a^b 2py sy \, dx, \quad \text{so}
\]

\[
0 = \delta \text{num} - \lambda \delta \text{den} = -2 \int_a^b \left[(ry')' - gy + \lambda py\right] sy \, dx
\]

\( sy \) is arbitrary so \( (ry')' - gy + \lambda py = 0 \).
Problem 1 (d)

The nth eigenfunction is

\[
In[1]:= f[x_, n_] := \frac{1}{x^2} \sin \left( \frac{n \pi \log[x]}{\log[3]} \right)
\]

We plot the first five of these after defining a function `graph[n]` which plots the nth one.

\[
In[2]:= graph[n_] := Plot[f[x, n], {x, 1, 3}, AxesLabel -> \{"x", "f[x]"\},
PlotLabel -> Row[\{"Problem 1 Assignment #6, \ "n = ", n\}]]
\]

We test the function graph for \(n = 3\).

\[
In[3]:= graph[3]
\]

It looks reasonable and we see the expected two interior nodes. Now we plot them all with a Do Loop. For clarity we show them on separate graphs.

\[
In[4]:= Do[Print[graph[i], \{i, 1, 5\}]]
\]
Problem 1 Assignment \( n = 1 \)

Problem 1 Assignment \( n = 2 \)

Problem 1 Assignment \( n = 3 \)
Here we plot the function \( f(x) = \frac{1}{x^2} \), and a partial sum of the expansion of the function in the eigenfunctions of the problem. For the series, we first define the nth coefficient, then the nth term and then a partial sum.

\[
\begin{align*}
\text{In[6]} &= c[n_] := \text{If}[\text{OddQ}[n], \frac{4}{(n \pi)}, 0] \\
\text{In[6]} &= \text{term}[x_, n_] := c[n] \frac{1}{x^2} \\
\text{In[7]} &= \text{partsum}[x_, k_] := \text{Sum}[\text{term}[x, n], \{n, 1, k\}]
\end{align*}
\]

Because the convergence is only like \( 1/n \), we will need a lot of terms. We try 100 terms, which should give us an error of the order of 1%. We define a function plotter[n] which plots the exact function in blue and the nth partial sum in red.

\[
\begin{align*}
\text{In[8]} &= \text{plotter}[n_] := \text{Plot}[\{\frac{1}{x^2}, \text{partsum}[x, n]\}, \{x, 1, 3\}, \text{PlotStyle} \rightarrow \{\text{Blue, Red}\}]
\end{align*}
\]

Now we try it for \( n = 100 \).
We try again with 200 terms.

The representation is generally good except at the endpoints, where the given function does not satisfy the boundary conditions satisfied by the eigenfunctions.

Challenge Problem

- (d)

We start by defining the trial function (as an expression). We call it yt

\[ yt = x(1 - x) + ax(1 - x)^2 \]

The derivative is ytprime:

\[ yt' = D[yt, x] \]

The numerator integral is
The denominator integral is

\[
\text{den} = \int_0^1 y t^2 \, dx
\]

\[
\frac{1}{30} + \frac{a}{70} + \frac{a^2}{630}
\]

We plot the functional \( F \) as a function of \( a \).

\[
\text{Plot}[\text{num} / \text{den}, \{a, -1, 3\}, \text{AxesLabel} \rightarrow \{"a", "F"\}]
\]

We see that there is a minimum around \( a = 1 \). We find the minimum.

\[
\text{ans} = \text{FindRoot}[\text{D[\text{num} / \text{den}, a] = 0, \{a, 1\}]
\]

\[
\{a \rightarrow 1.13314\}
\]

We calculate the value of \( F \) for this value of \( a \). This will be our best estimate of the eigenvalue for a trial function of this form.

\[
\text{eigest} = \text{num} / \text{den} /. \text{ans}
\]

\[
9.86975
\]

We compare this with \( \pi^2 \):

\[
\text{trueig} = \text{N[\pi^2, 6]}
\]

\[
9.86960
\]

The percentage error is

\[
\frac{100 (\text{eigest} - \text{trueig})}{\text{trueig}}
\]

\[
0.00147139
\]

An error of only about 1.5 in the fifth digit!