(1) Set Mathematica Notebook.

(2) The only tool we have for analyzing this incompletely specified problem is the diffusion time, which we discussed in class for a slab. We showed that the time required for appreciable cooling of a slab of width \( L \) is

\[
\tau = \frac{L^2}{\pi^2 D}
\]

where \( D \) is the thermal diffusivity of the slab material. A more precise characterization of this time is the e-folding decay time for the first mode in the slab.

In extending this to the sphere, a geometry we have not yet analyzed, we have to decide whether to use the factor of \( \pi^2 \) and we have to decide whether to use the sphere radius or sphere diameter for \( L \). The smallest value of \( \tau \) occurs when we take \( L = \text{radius} \) and keep the \( \pi^2 \):

\[
\tau_{\min} = \frac{(5 \times 10^{-2})^2}{\pi^2 (1.47 \times 10^{-6})} = 150 \text{ s}.
\]

The largest value occurs for \( L = \text{diameter} \), with \( \pi^2 \) omitted:

\[
\tau_{\max} = \frac{(8 \times 10^{-2})^2}{1.47 \times 10^{-6}} = 4354 \text{ s}.
\]

This is a very large range, but (clearly) 1.5 s is too small and 150,000 s is too large. We will work with the colleague who predicted 150 s. (A mathematical analysis of radially symmetric modes in the sphere shows that \( \tau_{\text{min}} \) is the correct e-folding time for the first mode.)
(3) Let $T_s(x)$ be the steady-state solution. Then
\[
\frac{d^2T_s}{dx^2} = -\frac{1}{D}, \quad 0 < x < L, \quad T_s(0) = 0, \quad T_s(L) = T_i.
\]
The transient part is then $\hat{T} = T - T_s$. By substituting $T = T + T_s$ into the equation and other conditions of the original process, we get
\[
\frac{\partial \hat{T}}{\partial t} = D \frac{\partial^2 \hat{T}}{\partial x^2}, \quad 0 < x < L, \quad t > 0
\]
with $\frac{\partial \hat{T}}{\partial t}(0, t) = 0$, $\frac{\partial \hat{T}}{\partial t}(L, t) = 0$, $\frac{\partial \hat{T}}{\partial t}(x, 0) = T_0 - T_s$.

(b) We integrate $\frac{d^2T_s}{dx^2} = -\frac{1}{D}$ twice to get
\[
T_s = -\frac{1}{2D}x^2 + Ax + B.
\]
Then $T_s(0) = 0 \Rightarrow B = 0$, $T_s(L) = T_i \Rightarrow T_i = -\frac{1}{2D}L^2 + AL$ so
\[
A = \frac{T_i}{L} + \frac{1}{2D}L.
\]
Then
\[
T_s(x) = \frac{1}{2D}x(L-x) + \frac{T_i x}{L}.
\]

(c) We will make use of results obtained in class. We showed in class that the separated solutions satisfying the equation and the homogeneous boundary conditions are
\[
e^{-\frac{\pi^2\nu^2}{L^2}} \sin \left( \frac{\nu \pi x}{L} \right), \quad \nu = 1, 2, 3, \ldots
\]
We construct $\hat{T}$ by superposing these:
\[
\hat{T}(x, t) = \sum_{\nu=3}^{\infty} C_n e^{-\frac{\pi^2\nu^2}{L^2}t} \sin \left( \frac{\nu \pi x}{L} \right).
\]
(3) (continued) We impose the initial condition on $T$

$$\hat{T}(x,0) = T_0 - T_s(x) = T_0 - \frac{k}{2D} x (L-x) + T_1 \frac{x}{L}$$

$$= \sum_{n=0}^{\infty} C_n \sin\left(\frac{n\pi x}{L}\right).$$

This is a Fourier sine series. We get

$$C_n = \frac{2}{L} \int_0^L [T_0 - \frac{k}{2D} x (L-x) + T_1 \frac{x}{L}] \sin\left(\frac{n\pi x}{L}\right) \, dx$$

We use Mathematica to evaluate the integral:

$$C_n = \frac{2}{\pi^2} \left[ (-1)^n T_0 + \frac{2}{\pi^2} \frac{L^2}{n^2} + \frac{2}{\pi^2} \frac{L^2}{(2n)^2} \right]$$

(d) We have $T(x,t) = T_s(x) + \sum_{n=0}^{\infty} C_n e^{-n^2 \frac{\pi^2}{L^2} t} \sin\left(\frac{n\pi x}{L}\right)$.

Then $\lim_{t \to \infty} T(x,t) = T_s(x) = \frac{k}{2D} x (L-x) + T_1 \frac{x}{L}$.

We see that $T_s(x)$ does not depend on the initial temperature $T_0$, so a knowledge of the steady-state temperature gives us no knowledge of the initial temperature. That information is lost completely in the limit $t \to \infty$.

(e) $\hat{T}(x,t) = C_1 e^{-\frac{n^2 \pi^2 t}{L^2}} \sin\left(\frac{n\pi x}{L}\right) + C_2 e^{-\frac{9n^2 \pi^2 t}{L^2}} \sin\left(\frac{3n\pi x}{L}\right)$

+ $C_3 e^{-\frac{9n^2 \pi^2 t}{L^2}} \sin\left(\frac{3n\pi x}{L}\right)$

The higher terms in the series have time exponentials which decay more rapidly. If $e^{-\frac{n^2 \pi^2 t}{L^2}} \ll e^{-\frac{9n^2 \pi^2 t}{L^2}}$, then the second
(3) (c) (continued) can be ignored compared with the first term, in which case

\[ f(x,t) \approx C_1 e^{-\frac{x^2}{4Dt}} \sin \left( \frac{\pi x}{L} \right). \]

The condition for this is

\[ e^{-\frac{x^2}{4D}} \ll 1 \]

or

\[ e^{-\frac{3L^2}{4Dt}} \ll 1. \]

We generally state this as

\[ \frac{\pi^2 D t}{L^2} \gg 1. \]

Thus the expanded time \( t \) should be as large or larger than the diffusion time.

(4) For \( |T - T_s| \leq 2^\circ C \), we are quite close to steady state. In that case, we can use the one-term approximation for the transient solution. Then our condition is

\[ \left| C_1 \right| e^{-\frac{x^2}{4D}} \sin \left( \frac{\pi x}{L} \right) \leq T_e, \quad t = 2^\circ C. \]

The maximum occurs at \( x = \frac{L}{2} \) where

\[ \sin \left( \frac{\pi x}{L} \right) = 1, \quad \text{so} \quad \left| C_1 \right| e^{-\frac{x^2}{4D}} \leq T_e. \]

From here we get

\[ t \geq \frac{L^2}{4D} \ln \left( \frac{|C_1|}{T_e} \right) \text{ dimensionless factor} \]

We will finish this problem with mathematics. We will also check our assumption that one term in the series for \( t \) is sufficient.
We try $T = F(x,t,G)$. We substitute this into the equation and divide by $DG$. We get
\[ \frac{1}{D_t G} \frac{DG}{Dt} = (1 + \beta t) \frac{1}{F} \frac{d^2 F}{dx^2}. \]

The separation hasn't quite worked, but it is easy to see how to fix it: divide by $D_t G$:
\[ \frac{1}{D_t G} \frac{DG}{Dt} = \frac{1}{F} \frac{d^2 F}{dx^2} = -\lambda. \]

We start with the $t$-equation:
\[ \frac{1}{G} \frac{DG}{Dt} = -\lambda D_t (J + \beta t). \]

We integrate with $t$:
\[ D_t G = -\lambda D_t (t + \frac{1}{2} \beta t^2) \quad \text{+ const}, \]
so
\[ G(t) = \text{constant} \cdot e^{-\lambda D_t (t + \frac{1}{2} \beta t^2)}. \]

Now the $x$-equation:
\[ \frac{d^2 F}{dx^2} + \lambda F = 0, \quad 0 < x < a, \quad f(0) = 0, \quad f(a) = 0. \]

We have solved this problem several times. The result is
\[ \lambda_n = \frac{n \pi^2}{a^2}, \quad F_n = \sin \left( \frac{n \pi x}{a} \right), \quad n = 1, 2, 3, \ldots \]

Apart from a multiplicative constant, our separate solutions are
\[ e^{-\frac{\lambda_n t}{2} D_t (1 + \frac{1}{2} \beta t)} \sin \left( \frac{n \pi x}{a} \right). \]
We take a superposition of these as the solution of the original problem:

\[ I(x,t) = \sum_{n=1}^{\infty} C_n e^{-\frac{n^2 \pi^2 \alpha}{a^2} t} \sin \left( \frac{n \pi x}{a} \right) \]

We impose the initial condition:

\[ I(x,0) = \frac{1}{a} \int_0^a x \left( 1 - \frac{x^2}{a^2} \right) \sin \left( \frac{n \pi x}{a} \right) \, dx = \sum_{n=1}^{\infty} \frac{C_n}{n} \sin \left( \frac{n \pi x}{a} \right) \]

This is a Fourier sine series, so

\[ C_n = \frac{2}{a} \int_0^a \frac{x}{a} \left( 1 - \frac{x^2}{a^2} \right) \sin \left( \frac{n \pi x}{a} \right) \, dx \]

\[ = \frac{(-1)^n n^4 \pi^2}{a^3} \]

\[ = \frac{(-1)^{n+1} 12 \pi^2}{n^3 a^3} \]  

So

\[ I(x,t) = \frac{12 \pi^2}{a^3} \sum_{n=1}^{\infty} (-1)^{n+1} \frac{e^{-\frac{n^2 \pi^2 \alpha}{a^2} t}}{n^3} \sin \left( \frac{n \pi x}{a} \right) \].
Problem 1

In[1]:= Clear[L, fbas, fext, fodd, foddext, feven, fevenext];

(a)

We begin by defining the base function and its periodic extension. We call the base function fbas, and the periodic extension fext.

In[2]:= fbas[x_] := 1 + x^2 - x^3

In[3]:= fext[x_] := fbas[Mod[x, 2*L, -L]]

In[4]:= L = 1;

We plot f over [-3,3].

In[5]:= Plot[fext[x], {x, -3, 3}, AxesLabel -> {"x", "fext(x)"}, AxesOrigin -> {0, 0}]

We calculate the Fourier coefficients. First the cosine coefficients.

In[6]:= a[0] = (1/2*L) Integrate[fbas[x], {x, -L, L}]


In[7]:= a[n_] = Simplify[(1/L) Integrate[fbas[x] Cos[(n π x)/L], {x, -L, L}], n ∈ Integers]

Out[7]= 4 (-1)^n / (n^2 π^2)
Now the sine coefficients.

```
In[8]:= b[n_] = Simplify[(1/L) Integrate[fbas[x] Sin[(n \[Pi] x) / L], {x, -L, L}, n \[Element] Integers]
Out[8]=
```

\[
\frac{2 (-1)^n \left( -6 + n^2 \pi^2 \right)}{n^3 \pi^3}
\]

We see that the sine coefficients drop off only like 1/n, and this is consistent with the fact that the extended function has discontinuities. The series will converge to fbas[x] on -1 < x < 1. At the discontinuities at ±1, the series converges to the mean of the right and left limits which is 2.

(b)

For the sine series, the function represented is the periodic extension of the odd extension. We first define the odd extension.

```
In[9]:= fodd[x_] := If[x > 0, fbas[x], -fbas[-x]]
```

Now we define the periodic extension of fodd. We call it foddext.

```
In[10]:= foddext[x_] := fodd[Mod[x, 2*L, -L]]
```

We plot foddext over [-3,3].

```
In[11]:= Plot[foddext[x], {x, -3, 3}, AxesLabel -> {"x", "foddext(x)"}, AxesOrigin -> {0, 0}]
```

As we saw earlier, we may eliminate the vertical segments (which are not really part of the graph) with the Option Exclusions:
Now we calculate the Fourier coefficients. Because \( f(x) \) is odd, only the sine coefficients are nonzero.

\[
\begin{align*}
\text{In}[13]:= & \quad b[n_] = \text{Simplify[(2/L) \text{Integrate}[\text{fbas}[x] \sin[(n \pi x)/L], \{x, 0, L\}, n \in \text{Integers}]} \\
\text{Out}[13]= & \quad -\frac{2 \left(2 + 4 (-1)^n + (-1)^n \ n^2 \pi^2\right)}{n^3 \pi^3}
\end{align*}
\]

We see that the coefficients drop off only as \( 1/n \), consistent with the fact that the extended function has discontinuities. The series converges to \( f_{\text{odd}}[x] \) for \(-1 < x < 0\) and \(0 < x < 1\). At the points of discontinuity \((-1,0,1)\) the series converges to the mean of the right and left limits, which in this case is zero.

\[
\text{(c)}
\]

For the cosine series, the function represented is the periodic extension of the even extension. We first define the even extension.

\[
\text{In}[14]:= \text{feven}[x_] := \text{If}[x > 0, \text{fbas}[x], \text{fbas}[-x]]
\]

Now we define the periodic extension of feven. We call it fevenext.

\[
\text{In}[15]:= \text{fevenext}[x_] := \text{feven[Mod[x, 2*L, -L]]}
\]

We plot fevenext over \([-3,3]\).
In[16]:= \text{Plot[f[	ext{evenext][x]}}\text{, \{x, -3, 3\}, AxesLabel \rightarrow \{"x", \"evenext\text{[x]}\}\}, AxesOrigin \rightarrow \{0, 0\}]}

\text{Out[16]=}

Now we calculate the Fourier coefficients. Because fext(x) is even, only the cosine coefficients are nonzero.

In[17]:= \text{a[0] = (1/L) Integrate[fbas[x], \{x, 0, L\}]}

\text{Out[17]= 13/12}

In[18]:= \text{a[n_] = Simplify[(2/L) Integrate[fbas[x] Cos[(n \pi x) / L], \{x, 0, L\}, n \in \text{Integers}]}\text{]}

\text{Out[18]=}

We see that the coefficients drop off like $1/n^2$, consistent with the fact that the extended function is continuous but has discontinuous derivatives. The series converges to even[x] for $-1 \leq x \leq 1$.

Problem 3

In[19]:= \text{Clear[L, \gamma, dif, T0, T1, coeff, c, Tc, term, Ttran, Ts, Trangraph, Trangraphmod, Tgraph];}

\[ n \] (c)

The Fourier sine coefficients for the transient solution are

In[20]:= \text{coeff =}

\text{Out[20]=}

Here we have used dif for the diffusivity because D is a reserved symbol in Mathematica.

We put this in a slightly more convenient form and assign the result to the function c[n]:

In[21]:= \text{c[n_] = Collect[coeff, \{T0, T1, \gamma\}]}\text{]}

\text{Out[21]=}
We start by entering all of the parameter values into *Mathematica*.

\[
\begin{align*}
\text{diff} &= 1.2 \times 10^{-4} (\text{m}^2/\text{s}) ; \\
T_0 &= 100 (\text{°C}) ; \\
T_1 &= 50 (\text{°C}) ; \\
L &= 0.05 (\text{m}) ; \\
L_0 &= 22.0 (\text{°C}/\text{s}) ; \\
T_c &= 2 (\text{°C}) ;
\end{align*}
\]

We showed on the handwritten solutions that in order to have the transient temperature less than Tc in magnitude the time t has to be greater than

\[
\text{In}[24]:= \left(\frac{L^2}{\pi^2 \text{diff}}\right) \log\left[\frac{\text{Abs}[c[1]]}{T_c}\right]
\]


Thus the critical time is 6.12 s. This is somewhat larger than the diffusion time, but of the same order:

\[
\text{In}[25]:= \frac{L^2}{\pi^2 \text{diff}}
\]

Out[25] = 2.11086

To check our use of the one-term approximation, we calculate and plot the transient solution versus x for several values of t, using 10 terms in the series. We call the transient Ttran[x,t].

\[
\text{In}[26]:= \text{term}[x_-, t_-, n_] := c[n] \text{Exp}\left[-\left(n^2 \pi^2 \text{diff} t\right)/L^2\right] \text{Sin}\left[(n \pi x)/L\right]
\]

\[
\text{In}[27]:= \text{Ttran}[x_-, t_-] := \text{Sum}[\text{term}[x, t, n], \{n, 1, 10\}]
\]

We first define a function which creates the graph for a given t.

\[
\text{In}[28]:= \text{trangraph}[t_\_] := \text{Plot}[\text{Ttran}[x, t], \{x, 0, L\}, \text{AxesLabel} \rightarrow \{"x (m)", "Ttran (°C)"\}]
\]

We try this at t = 4 seconds.

\[
\text{In}[29]:= \text{trangraph}[4]
\]

We see that the transient will approach the value +2 from above, so the relevant reference line is at 2°C. We include this line in our plot.

\[
\text{In}[30]:= \text{trangraphmod}[t_\_] := \text{Plot}[\{T_c, \text{Ttran}[x, t]\}, \{x, 0, L\}, \text{PlotLabel} \rightarrow \text{Row}[\{"t = ", t, " s"\}], \\
\text{PlotStyle} \rightarrow \{\text{Red, Black}\}, \text{PlotRange} \rightarrow \{-1.0, 25.0\}, \text{AxesLabel} \rightarrow \{"x (m)", "Ttran (°C)"\}]
\]
Now we do a sequence of graphs with a Do loop.

\text{In[32]} = \text{Do[Print[trangraphmod[t]], \{t, 1, 7\}]}
We see that the critical time is between 6 and 7 s, and considerably closer to 6 s. We now refine our sequence and magnify the graph scale. To save a little space, we make the graphs smaller and display them two to a line by using the commands Graphics-Grid and Table.

```
In[33]:= transgraphmod[t_] := Plot[[Tc, Ttran[x, t]], {x, 0, L}, PlotRange -> {{0.01, 0.04}, {1.9, 2.1}}, PlotLabel -> Row["t = ", t, " s"], Ticks -> {{0.01, 0.02, 0.03, 0.04}, {2.0}}, ImageSize -> 250, PlotStyle -> {Red, Black}, AxesLabel -> {"x (m)", "Ttran (°C)"}]
```
The graph sequence shows that the critical time is 6.12 s, in agreement with the result found with the one-term approximation for the transient.

(g)

The transient solution, based on 10 terms of the series, was defined earlier as Ttran[x,t]. We define here the steady-state solution Ts[x], and the temperature T[x,t].

\[
\text{Ts}[x_] := \left(\frac{\gamma}{(2 \ \text{dif})}\right) x (L - x) + T_1 (x / L)
\]

\[
\text{T}[x_, t_] := \text{Ttran}[x, t] + \text{Ts}[x]
\]
Now we define Tgraph[t], a function which produces a graph of the temperature versus x at the specified time t. The graph also shows the initial condition dashed, and the final steady state in blue. We give a special definition for t=0, using the specified initial condition.

In[37]: Tgraph[t_] := Plot[{T[x, t], T0, Ts[x]}, {x, 0, L}, PlotStyle -> {Black, Dashed, Blue}, AxesLabel ->{"x (m)" , "T(x,t)", PlotRange -> (0, 110), AxesOrigin -> (0, 0), ImageSize -> 250, PlotLabel -> Row["t = ", PaddedForm[t, {3, 1}], " (s)"]}

In[38]: Tgraph[0.0] := Plot[{T0, Ts[x]}, {x, 0, L}, PlotStyle -> {Black, Blue}, AxesLabel ->{"x (m)" , "T(x,t)", ImageSize -> 250, PlotRange -> (0, 101), AxesOrigin -> (0, 0), PlotLabel -> "t = 0.0 (s)"}

Again we use the command GraphicsGrid and Table to construct the desired graphs, two to a line.

In[39]: GraphicsGrid[Table[{Tgraph[0.0 + i], Tgraph[0.0 + (i + 1)]}, {i, 0, 10, 2}]]
We see that to within graphical accuracy, we are essentially at steady state after about 9 or 10 seconds. We can also produce the sequence by using Manipulate, although that leaves less to see in the printed notebook.
If we want a really smooth movie of the transient process, we can use a Do loop to make many frames with a small time interval between them. We also need to use more terms in the transient series for small times. We carry this out, starting by redefining the transient to use a number of terms dependent on the time.

```math
\text{In[41]}: \quad \text{Ttran}[x\_, t\_] := \text{Module}[\{\text{nterms}, n\}, \\
\quad \text{If}[(t < 0.5), (\text{nterms} = 50), (\text{nterms} = 10)]; \text{Sum}[\text{term}[x, t], \{n, 1, \text{nterms}\}]]
```

```math
\text{In[42]}: \quad \text{Tgraph}[t\_] := \text{Plot}[\{\text{T}[x, t], \text{T0}, \text{T}[x]\}, \{x, 0, 1\}, \text{PlotStyle} \to \{\text{Black, Dashed, Blue}, \text{AxesLabel} \to \{"x (m)"}, "T(x,t)"\}, \text{PlotRange} \to \{0, 110\}, \text{AxesOrigin} \to \{0, 0\}, \text{ImageSize} \to 350, \text{PlotLabel} \to \text{Row}[\{"t = ", \text{PaddedForm}[t, \{4, 2\}], " (s)\}]]
```

```math
\text{In[43]}: \quad \text{Tgraph}[0.0] := \text{Plot}[\{\text{T0}, \{x, 0, 1\}, \text{PlotStyle} \to \{\text{Black, Blue}, \text{AxesLabel} \to \{"x (m)"}, "T(x,t)"\}, \text{PlotRange} \to \{0, 110\}, \text{AxesOrigin} \to \{0, 0\}, \text{ImageSize} \to 350, \text{PlotLabel} \to \text{Row}[\{"t = ", \text{PaddedForm}[0.0, \{4, 2\}], " (s)\}]]
```

```math
\text{In[44]}: \quad \text{Do}[\text{Print}[\text{Tgraph}[i*0.05]], \{i, 0., 300.\}]
```

If we want a really smooth movie of the transient process, we can use a Do loop to make many frames with a small time interval between them. We also need to use more terms in the transient series for small times. We carry this out, starting by redefining the transient to use a number of terms dependent on the time.
In the printed version of this notebook, only the first graph of the sequence is shown. In the *Mathematica* notebook, you can play a movie of the graph sequence by going to the Menu choice Graphics-> Rendering-> Animate Selected Graphics

### Challenge Problem

We calculate here the Fourier sine coefficients for the given initial condition.

\[
\text{In}[45]:= \quad \text{f}[	ext{x}_\_] := T2 \ (x / a) \ \left( 1 - (x / a)^2 \right)
\]

\[
\text{In}[46]:= \quad \text{b}[n\_] = \frac{2}{a} \ \text{Simplify[\text{Integrate[f[x] Sin[(n \pi x) / a], \{x, 0, a\}], n \in \text{Integers}]}}
\]

\[
\text{Out}[46]= \quad -\frac{12 (-1)^n T2}{n^3 \pi^3}
\]