(1) (a) \( \nabla \cdot F = \frac{\partial}{\partial x} (3y^2) + \frac{\partial}{\partial y} (-2x^2) + \frac{\partial}{\partial z} (2z^2) = 2z \)

\( \nabla \cdot G = \frac{\partial}{\partial x} (2x-y+3y^2) + \frac{\partial}{\partial y} (-x-2x^2-yz) + \frac{\partial}{\partial z} (y+2z+2z^2) = 2-4+2+2z = 2z \)

(b) By the divergence theorem, \( \iint_S F \cdot n \, dS = \iiint_V \nabla \cdot F \, dV \)

And \( \iint_S G \cdot n \, dS = \iiint_V \nabla \cdot G \, dV \). Because \( \nabla \cdot F = \nabla \cdot G \), the integrals are equal.

(2) (a) \( \nabla \times F = \left( -\frac{\partial F_y}{\partial z} + \frac{\partial F_z}{\partial y} \right) i + \left( \frac{\partial F_z}{\partial x} - \frac{\partial F_x}{\partial z} \right) j + \left( \frac{\partial F_x}{\partial y} - \frac{\partial F_y}{\partial x} \right) k \)

\( \nabla \times G = \left( -\frac{\partial G_y}{\partial z} + \frac{\partial G_z}{\partial y} \right) i + \left( -\frac{\partial G_z}{\partial x} + \frac{\partial G_x}{\partial z} \right) j + \left( \frac{\partial G_x}{\partial y} - \frac{\partial G_y}{\partial x} \right) k \)

So \( \nabla \times (F - G) = 0 \).

(b) Because \( \nabla \times (F - G) = 0 \), there exists a \( \Phi \) such that \( F - G = \nabla \Phi \), hence \( F = G + \nabla \Phi \).

(c) \( \nabla \Phi = F - G = (2x+y) i + (x+xy-z-j) j - (y+2z) k \)

Then \( \frac{\partial \Phi}{\partial x} = 2x+y \), \( \frac{\partial \Phi}{\partial y} = x+xy-z \), \( \frac{\partial \Phi}{\partial z} = -y-2z \).

We integrate the first equation to get
\( \Phi = -x^2 + xy + f(y,z) \).
(2) (c) (continued) Substituting this result in the
equation for \( \partial \Phi/\partial y \) we get

\[
x + \frac{\partial f}{\partial y} = x + xy - z
\]

so

\[
\frac{\partial f}{\partial y} = xy - z
\]

\[
f(y, z) = 2y^2 - 2y + 5(z)
\]

So \( \Phi = -x^2 + xy + 2y^2 - 2y + 5(z) \). We
substitute this into the equation for \( \partial \Phi/\partial z \):

\[
-x + 5y' = xy - 2z \quad \text{so}\quad y'(z) = -2z
\]

\[
g(z) = -z^2 + 100\pi z^2
\]

The final result for \( \Phi \) is

\[
\Phi = -x^2 + xy + 2y^2 - 2y - z^2 + 100\pi z^2
\]

The constant is arbitrary.

Check:

\[
\Phi + \nabla \Phi = (2x - y + 3y^3) \hat{i} + (-x - 2x^2 + y + 2z) \hat{j} + (y + 2z + 2) \hat{k}
\]

\[
+ (-2x + y) \hat{i} + (x + 4y - z) \hat{j} + (-y - 2z) \hat{k}
\]

\[
= 3y^2 \hat{i} - 2x^2 \hat{j} + 2 \hat{k} = F.
\]

(3) We have \( F = \Phi + \nabla \Phi \), so \( \int_C F \cdot ds = \int_C \Phi \cdot ds \).

\[
\nabla \Phi \cdot ds = d\Phi, \quad \text{so} \quad \int_C \nabla \Phi \cdot ds = \int_C d\Phi = \Phi|_P - \Phi|_P. \quad \Phi
\]

is quadratic in the coordinates and so has the
same value \( C \) at \( (1, 1, 1) \) as at \( (-1, -1, -1) \). i.e.

\[
\int_C F \cdot ds = \int_C G \cdot ds.
\]
The temperature seen by the plane is

$$T(t) = T(x(t), y(t), z(t))$$

where $x$, $y$, $z$ are the position coordinates of the plane. The time rate of change of this is

$$\frac{dT}{dt} = \frac{\partial T}{\partial x} \frac{dx}{dt} + \frac{\partial T}{\partial y} \frac{dy}{dt} + \frac{\partial T}{\partial z} \frac{dz}{dt}$$

$$= a V_x + b V_y + c V_z$$

where $V_x$, $V_y$, $V_z$ are the velocity components of the plane. If $d$ is the angle of climb, then $V_z = V_0 \sin d$, and the projection of the velocity onto the $x-y$ plane is $V_0 \cos d$. Then

$$V_x = V_0 \cos (\cos d)(\cos d), V_y = V_0 \sin (\cos d)(\cos d).$$

So

$$\frac{dT}{dt} = \frac{a}{V_z} V_0 \cos d + \frac{b}{V_z} V_0 \cos d - (V_0 \sin d)$$

$$= \frac{V_0 \cos d}{V_z} (a + b) - V_0 \sin d (c)$$

We are given that $\frac{dT}{dt} = 1.47^\circ/\text{minute}$. We have $V_0 = 300 \text{ km/hr} = 5 \text{ km/minute}$, so

$$1.47 = \frac{5}{V_z} (a + b) - 5 \sin d (c)$$

We can either use trial and error with a calculator or use finding technique such as FindRoot in Mathematica to get $d = 8.06$ degrees.

For zero rate of change, we need

$$0 = 1.7678 \cos d - 2 \sin d$$

or $\tan d = 0.8839$

$$\Rightarrow d = 41.5 \text{ degrees}.$$
(5) For this inhomogeneous linear first-order equation, an integrating factor is the most efficient technique. According to the basic theory, the factor in this case is \( e \int 2 \, dt = e^{2t} \).

We multiply through by this to get

\[ e^{2t} \frac{dx}{dt} + 2te^{2t} x = 2t \]

\[ \frac{d}{dt} (e^{2t} x) = 2t. \]

\[ e^{2t} x(t) - e^{2t} x(0) = \frac{t^2}{2} \]

\[ e^{2t} x(t) - 3 = \frac{t^2}{2} \]

\[ x(t) = (3 + \frac{t^2}{2}) e^{-2t}. \]

(6) For second-order, linear, homogeneous constant-coefficient equations, the form of the solution is \( x = e^{rt} \). We substitute this into the equation and (with) the common factor of \( e^{rt} \) to get

\[ r^2 + 4r + 20 = 0 \]

Which gives \( r = -2 \pm \sqrt{16} \). Therefore, we can use the solution basis \( e^{(2 \pm \sqrt{16})t} \).

It is more convenient to use the real-valued basis \( e^{-2t} \cos(4t), e^{-2t} \sin(4t) \). So \( x(t) = y e^{-2t} \cos(4t) + z e^{-2t} \sin(4t) \).

We determine \( C_1 \) and \( C_2 \) by imposing the initial conditions:

\[ x(0) = 1 = C_1, \quad \frac{dx}{dt}(0) = 2 = -2C_1 + 4C_2 \Rightarrow C_1 = 1, C_2 = 1. \]

\[ x(t) = e^{-2t} (\cos(4t) + \sin(4t)). \]

(7) Try \( y = e^{3t} \). Substitute into the equation to get \( r = 6 \).

The solution basis is then \( (\cos(3t), \sin(3t)) \), so \( y = C_1 \cos(3t) \).

We impose the initial conditions:

\[ y(0) = 0 = C_1, \quad y'(0) = 6 = 3C_2 \Rightarrow C_1 = 0, C_2 = 2, \quad y(t) = 2 \sin(3t). \]
(8) We try \( y = e^x \). Substituting into the equation gives
\[
\frac{d^2y}{dx^2} + 2\frac{dy}{dx} - 3 = 0 \Rightarrow r = -3, 1,
\]
so \( y(x) = c_1 e^{-3x} + c_2 e^x \).

We impose the initial conditions: \( y(0) = 0 = c_1 + c_2 \) and \( y'(0) = 2 = -3c_1 + c_2 \). We solve for \( c_1 \) and \( c_2 \) to get \( c_1 = -\frac{1}{2}, c_2 = \frac{1}{2} \) and \( y(x) = \frac{1}{2}(e^x - e^{-3x}) \).

(9) This linear equation is of a special form called an \textit{equi-dimensional} equation. The name comes from the fact that every term has the same dimensionality in \( x \). What you learn in basic differential equations is that such equations have solutions of the form \( y = x^n \). We substitute this into the equation and then cancel the common factor of \( x^n \) to get
\[
(n(n-1))x^{n-2} - 3 = 0
\]
so \( n = -3, 1 \). The solution is \( y = c_1 x + c_2 x^{-3} \).

\textbf{Challenge Problem}

If \( \nabla \cdot F = 0 \), then there is a \( \Phi \) such that \( F = \nabla \Phi \).
If in addition \( \nabla \times F = 0 \), then \( \nabla \cdot (\nabla \Phi) = \nabla^2 \Phi = 0 \).
The equation \( \nabla^2 \Phi = 0 \) is Laplace's equation, and we will spend considerable time in the course learning how to solve it. What we have shown here is that the search for a vector field with zero (\( \nabla \) or \( \nabla \times \)) and divergence can be transformed to the search for solutions of Laplace's equation. It is easy to see that any linear function is a solution of Laplace's equation. However the gradient of a linear function is a constant vector, and we are looking for non-constant vector fields with zero (\( \nabla \) or \( \nabla \times \)) and divergence. Let's try a quadratic.
Challenge Problem (continued)

Function: \( \Phi = \alpha x^2 + \beta y^2 + \gamma z^2 \). Then

\[ \nabla^2 \Phi = 2\alpha + 2\beta + 2\gamma \]

So we require \( \alpha + \beta + \gamma = 0 \). We take \( \alpha = 1, \beta = 1, \gamma = -2 \)

Hence

\[ \Phi = x^2 + y^2 - 2z^2 \]

and

\[ F = \nabla \Phi = 2x \hat{i} + 2y \hat{j} - 4z \hat{k} \]

But \( F \) is non-constant, and \( \nabla \cdot F = 2 + 2 - 4 = 0 \), and

\[ \nabla \times F = \left( \frac{\partial}{\partial y} (-4z) - \frac{\partial}{\partial z} (2y) \right) \hat{i} + \left( \frac{\partial}{\partial z} (2x) - \frac{\partial}{\partial x} (-4z) \right) \hat{j} + \left( \frac{\partial}{\partial x} (2y) - \frac{\partial}{\partial y} (2x) \right) \hat{k} = 0. \]