Assignments handed in by 6 PM on Thursday Oct. 14 will receive a 5-point bonus. Assignments handed in after that but by 4 PM on Friday Oct. 15 will receive full credit but no bonus. No assignments will be accepted after 4 PM on Oct. 15.

### LECTURE SCHEDULE AND READING

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### PROBLEMS

#### 3.2 LAPLACE AND LAPLACE-LIKE EQUATIONS

(1) (30 points) Consider the boundary value problem given below for the steady-state temperature $T$ in a rectangle. The quantity $T_0$ appearing in a boundary condition is a constant.

$$\frac{\partial^2 T}{\partial x^2} + \frac{\partial^2 T}{\partial y^2} = 0, \quad 0 < x < a, \quad 0 < y < b,$$

with $\frac{\partial T}{\partial y}(x,0) = 0$, $\frac{\partial T}{\partial x}(0,y) = 0$, $\frac{\partial T}{\partial x}(a,y) = 0$, and $T(x,b) = 4T_0 \frac{x}{a} \left(1 - \frac{x}{a}\right)$.

(a) (10 points) Solve this problem by separation of variables and superposition. For parts (b) and (c) below, use Mathematica and use the values $a = 2$ m, $b = 1$ m, and $T_0 = 100$ °C.

(b) (5 points) Check your calculations by comparing selected numerical values of the boundary temperature on $y = b$ with values calculated from your series. As a further check on your work, show from your solution that the temperature at the midpoint of the rectangle is 75.04 °C.

(c) (10 points) Plot lines of constant temperature for 50, 60, 70, 80, and 90 °C on a single contour plot. Use the Option ContourLabels -> All to get printed labels on each contour. Construct a second contour plot either for the upper left or upper right corner of the rectangle to show the temperature contours for 10, 20, 30, 40, 50, 60, and 70 °C.

(d) (5 points) What is the total heat flow into or out of the top boundary at $y = b$? (Hint: You should be able to answer this without doing any calculations.)

(continued on next page)
(2) (25 points) Use separation of variables to solve the boundary value problem given below for a Laplace-like equation in a rectangle.

\[
\frac{\partial^2 \Phi}{\partial x^2} + \frac{\partial^2 \Phi}{\partial y^2} = \Phi, \quad 0 < x < 1 \text{ and } 0 < y < 2,
\]

with \( \Phi(x,0) = 0, \Phi(x,2) = 0, \frac{\partial \Phi}{\partial x}(0,y) = 0, \) and \( \Phi(1,y) = 1. \)

3.3, 3.4 WAVE EQUATION

(3) (20 points) Consider the initial value problem, given below, for the wave equation with given initial displacement and initial velocity.

\[
\frac{\partial^2 y}{\partial t^2} = c^2 \frac{\partial^2 y}{\partial x^2}, \quad 0 < x < L, \quad t > 0, \quad \text{with } y(0,t) = 0, \quad y(L,t) = 0,
\]

\[y(x,0) = y_0 \sin \left( \frac{\pi x}{L} \right), \quad \text{and} \quad \frac{\partial y(x,0)}{\partial t} = v_0 \sin \left( \frac{\pi x}{L} \right).\]

Here \( y_0 \) and \( v_0 \) are positive constants.

(a) (15 points) Use the separated solutions found in class to solve this initial value problem.

(b) (5 points) Find the maximum displacement of the string.

3.5 ENERGY INTEGRALS AND UNIQUENESS

(4) (25 points) In this problem you will use an energy integral to prove uniqueness for the boundary value problem given below for the heat equation. \( V \) is a finite volume bounded by a smooth surface \( S. \)

\[\frac{\partial T}{\partial t} = \nabla^2 T + \Gamma(r,t) \text{ in } V, \text{ with } T_S = f(r,t), \text{ and } T(r,0) = g(r).\]

Here \( \Gamma, f \) and \( g \) are known functions, and the diffusivity \( D \) is a known positive constant.

(a) (5 points) Let \( T_1 \) and \( T_2 \) be two solutions of the above problem. Formulate the problem for the difference \( \hat{T} = T_1 - T_2. \)

(b) (10 points) Introduce the “energy” integral \( E = \iiint_V \frac{1}{2} \hat{T}^2 \, d\tau . \) Show that \( \frac{dE}{dt} \leq 0. \)

(c) (5 points) Prove that the solution of the boundary value problem is unique.

CHALLENGE PROBLEM

In part (a) of this problem you will develop a general technique for solving the forced wave equation, and in part (b) you will apply your result of part (a) to a simple example. The
The problem to be solved is given by
\[ \frac{\partial^2 y}{\partial t^2} = c^2 \frac{\partial^2 y}{\partial x^2} + p(x,t), \text{ with } y(0,t) = 0 \text{ and } y(L,t) = 0, \]
and with \( y(x,0) = f(x) \), and \( \frac{\partial y}{\partial t}(x,0) = g(x) \).

The new feature here is the given applied force \( p \). (More precisely, if \( P(x,t) \) is the force applied per unit length at location \( x \) and time \( t \), then \( p(x,t) = P(x,t)/\sigma \), where \( \sigma \) is the mass per unit length of the string.) The initial displacement \( f(x) \) and the initial velocity \( g(x) \) are given functions. Because of the force \( p \), a direct attack by separation of variables will not work. A new technique which will work may be described informally as expanding everything in sight in a Fourier sine series:

\[ p(x,t) = \sum_{n=1}^{\infty} p_n(t) \sin(n\pi x / L), \quad y(x,t) = \sum_{n=1}^{\infty} y_n(t) \sin(n\pi x / L), \]
\[ f(x) = \sum_{n=1}^{\infty} f_n \sin(n\pi x / L), \quad g(x) = \sum_{n=1}^{\infty} g_n \sin(n\pi x / L). \]

Let’s take a minute to consider what is known and what is to be determined. Because \( p(x,t) \) is a known function, the coefficients \( p_n(t) \) are known. Because the initial functions \( f \) and \( g \) are known, the coefficients \( f_n \) and \( g_n \) are known. Solving the problem then requires us to find the unknown coefficients \( y_n(t) \).

(a) Substitute the expansions for \( y \) and \( p \) into the partial differential equation. (Note that our license for differentiating the series for \( y \) comes from the fact that both \( y \) and the eigenfunctions \( \sin(n\pi x / L) \) satisfy the same boundary conditions.) Show that the result of the substitutions is a second-order, linear, inhomogeneous ordinary differential equation for each \( y_n(t) \). Find from the expansions of \( f \) and \( g \) the initial conditions for each \( y_n \). Using the method of variation of parameters from the basic theory of ordinary differential equations, one can derive an explicit solution of the problem for \( y_n(t) \). This is the subject of the bonus problem given below, but you are not required to do it here.

(b) Find the solution when \( p(x,t) = Ae^{-\alpha t} \sin(\pi x / L), f(x) = 0, \) and \( g(x) = 0. \)

**BONUS CHALLENGE PROBLEM**

If you do this bonus problem correctly, you will receive 15 extra points on the Challenge Problem.

Use variation of parameters to construct the solution for \( y_n(t) \) in the general case of part (a) above.