Assignments handed in by 6 PM on Wednesday September 22 will receive a 5 point bonus. Assignments handed in after that but by 6 PM on Thursday September 23 will receive full credit but no bonus. No assignments will be accepted after 6 PM on Thursday.

**LECTURE SCHEDULE AND READING**

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**PROBLEMS**

**2.2 CONVERGENCE OF FOURIER SERIES**

(1) (20 points) This problem is concerned with the Fourier series of \( f(x) = 7x^2 - 5x^4 + x^6 \) on \(-1 \leq x \leq 1\).

(a) (5 points) Either sketch by hand or use Mathematica to plot on \(-3 \leq x \leq 3\) the periodically extended function represented by the Fourier series.

(b) (5 points) Based on the smoothness of the periodically extended function, make a prediction about how fast the coefficients drop off with \( n \) in the Fourier series for this function.

(c) (10 points) By calculating the Fourier coefficients (you might want to use Mathematica to do this), show that the Fourier series for \( f(x) \) on \(-1 \leq x \leq 1\) is given by

\[
f(x) = \frac{31}{21} + 1440 \sum_{n=1}^{\infty} \frac{(-1)^n}{n^6 \pi^6} \cos(n\pi x).
\]

Use your answer to check your prediction of part (b).

(2) (15 points) This problem deals with termwise differentiation of Fourier series.

(a) (10 points) Find the Fourier series for \( g(x) = 14x - 20x^3 + 6x^5 \) on \(-1 \leq x \leq 1\) by calculating the coefficients directly from the defining formulas. (Again you can save considerable labor by using Mathematica.)

(b) (5 points) Find the Fourier series for \( g(x) \) by differentiating termwise the series obtained in problem 1 for \( f(x) = 7x^2 - 5x^4 + x^6 \), and show that you get the same result as in part (a). (Note that \( f'(x) = g(x) \).)

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(3) (20 points) This problem deals with termwise integration of a Fourier series. The functions $f$ and $g$ here are the same as in problems 1 and 2, and you will use the Fourier series derived there in this problem. From basic calculus, we have $\int_{0}^{x} g(x')\,dx' = f(x)$ for any $x$ in $[-1,1]$, where we have used the fact that $f(0) = 0$ for this $f$. According to a theorem stated in class on integration of Fourier series, we get a valid result when we integrate the series for $g(x)$ termwise. Use the series you obtained for $g(x)$ to carry out the above integration. Carry out such an integration from $x = 0$ to a generic $x$. According to the above formula, the result of that integration should be the Fourier series for $f(x)$. Explore this in detail and show that the truth of that statement requires the following:

$$\sum_{n=1}^{\infty} \frac{(-1)^{n+1}}{n^6} = \frac{31\pi^6}{30240}.$$  

Use the sum function in Mathematica to verify the truth of this last equality. (It is possible to use this series to calculate approximations to $\pi$.)

2.3 ORTHOGONALITY

(4) (20 points)

(a) (5 points) Show that the three vectors given below (where $\mathbf{i}, \mathbf{j},$ and $\mathbf{k}$ are the usual rectangular Cartesian unit vectors) are linearly independent. Show also that no two vectors of the set are orthogonal.

$$\mathbf{E}_1 = \mathbf{i}, \quad \mathbf{E}_2 = \mathbf{i} + \mathbf{j}, \quad \mathbf{E}_3 = \mathbf{i} + \mathbf{j} + \mathbf{k}.$$  

(b) (5 points) Because the vectors are linearly independent, they form a basis for the space of three-dimensional vectors. Given the vector $\mathbf{A} = 3\mathbf{i} + 2\mathbf{j} + 4\mathbf{k}$, find constants $c_1$, $c_2$, and $c_3$ such that $\mathbf{A} = c_1 \mathbf{E}_1 + c_2 \mathbf{E}_2 + c_3 \mathbf{E}_3$.

(c) (5 points) Show that the three vectors given below are linearly independent. Show also that the vectors are pairwise orthogonal.

$$\mathbf{E}_1 = \mathbf{i} - 2\mathbf{j}, \quad \mathbf{E}_2 = 2\mathbf{i} + \mathbf{j} + \mathbf{k}, \quad \mathbf{E}_3 = -2\mathbf{i} - \mathbf{j} + 5\mathbf{k}.$$

(d) (5 points) Using the same $\mathbf{A}$ as in part (b), find constants $d_1$, $d_2$ and $d_3$ such that $\mathbf{A} = d_1 \mathbf{E}_1 + d_2 \mathbf{E}_2 + d_3 \mathbf{E}_3$.

(5) (20 points) In class we discussed Parseval’s theorem, which says that if $f(x)$ has, on the interval $[-L, L]$, the Fourier series

$$a_0 + \sum_{n=1}^{\infty} a_n \cos \left( \frac{n\pi x}{L} \right) + b_n \sin \left( \frac{n\pi x}{L} \right),$$

then $\int_{-L}^{L} |f(x)|^2 \, dx = 2L a_0^2 + \sum_{n=1}^{\infty} (|a_n|^2 + |b_n|^2)$.

We didn’t establish the conditions for which this is true, but it is known that a sufficient condition is that $f(x)$ is piecewise continuous on $[-L, L]$ (the theorem is actually true for a much more general class of functions). It is true whether $f$ is real or complex. If $f$ is real, all of the squares of absolute values become just squares.

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(5) (continued) (a) (10 points) The Fourier series on $-1 \leq x \leq 1$ of $g(x) = 14x - 20x^3 + 6x^5$ was obtained in problem (2). Use this series and Parseval’s theorem to prove that
\[ \sum_{n=1}^{10} \frac{1}{n^{10}} = \frac{\pi^{10}}{93555}. \]

(b) (5 points) Use Mathematica and the series of part (a) to calculate a numerical approximation to $\pi$. Find the minimum number of terms required to reproduce the digits 3.1415926536.

(c) (5 points) Use the Sum function in Mathematica to see if Mathematica knows the exact result for this series.

CHALLENGE PROBLEM

In this problem you will explore some of the mathematics behind Parseval’s Theorem. The context here will be more general, in that we will consider arbitrary sets of real orthogonal functions. A useful reference for this problem is Chapter 6 of *Fourier Analysis and Boundary Value Problems*, E. A. González-Velasco, Academic Press, 1995, San Diego (one of the course references on reserve in Carlson).

Suppose we have a set of functions $\phi_k(x)$, $k = 1, 2, \ldots$ defined on $a \leq x \leq b$. In what follows, you may assume these functions to be well-behaved (e.g., $C^2$). Suppose that the functions have the following properties: they are orthogonal
\[ \int_{a}^{b} \phi_m(x) \phi_n(x) \, dx = 0 \text{ for } m \neq n, \]
and they are normalized
\[ \int_{a}^{b} (\phi_m(x))^2 \, dx = 1. \]
Given a real integrable function $f(x)$ on $[a, b]$, we may calculate the Fourier coefficients with respect to the $\phi$'s:
\[ f_m = \frac{\int_{a}^{b} f(x) \phi_m(x) \, dx}{\int_{a}^{b} (\phi_m(x))^2 \, dx} = \int_{a}^{b} f(x) \phi_m(x) \, dx. \]
In class, we talked quite a bit about the pointwise convergence of the Fourier series for $f$ — that is the question of the convergence of $\sum_{n=1}^{\infty} f_n\phi_n(x)$ at each $x$, and we had a very specific theorem for piecewise smooth functions. In this problem, we will be talking about a different kind of convergence called convergence in the mean. Let the $N$th partial sum be $S_N(x) = \sum_{n=1}^{N} f_n\phi_n(x)$.

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Then we say that $S_N(x)$ converges in the mean to $f(x)$ if

$$\lim_{N \to \infty} \int_a^b (f(x) - S_N(x))^2 \, dx = 0.$$ 

An alternative notation for this is $\lim_{N \to \infty} S_N(x) = f(x)$. One last definition. We call the set $\{ \phi_n \}$ complete on $[a,b]$ if $\int_a^b f(x) \phi_n(x) \, dx = 0$ for all $n$ implies $\int_a^b [f(x)]^2 \, dx = 0$. That is, the set is complete if the only function orthogonal to every member of the set is a function with zero norm.

**a** (35 points) Show that for any set of orthonormal functions and any integrable function $f$, we have

$$\sum_{n=1}^{\infty} (f_n)^2 \leq \int_a^b [f(x)]^2 \, dx.$$ 

This is called Bessel’s inequality. With equality in the above relation, we would have Parseval’s theorem.

**b** (35 points) Show that $\lim_{N \to \infty} S_N(x) = f(x)$ if and only if Parseval’s theorem holds for $f$.

**c** (30 points) Show that if Parseval’s Theorem holds for every integrable $f$ on $[a,b]$, then $\{ \phi_n \}$ is complete on $[a,b]$. 