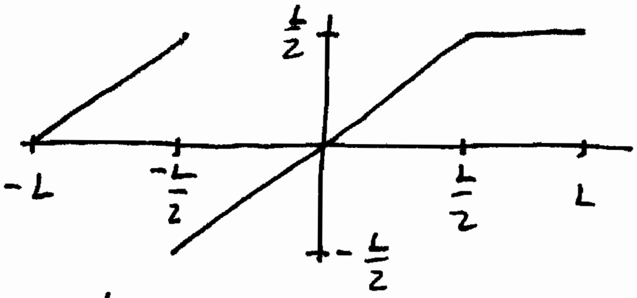
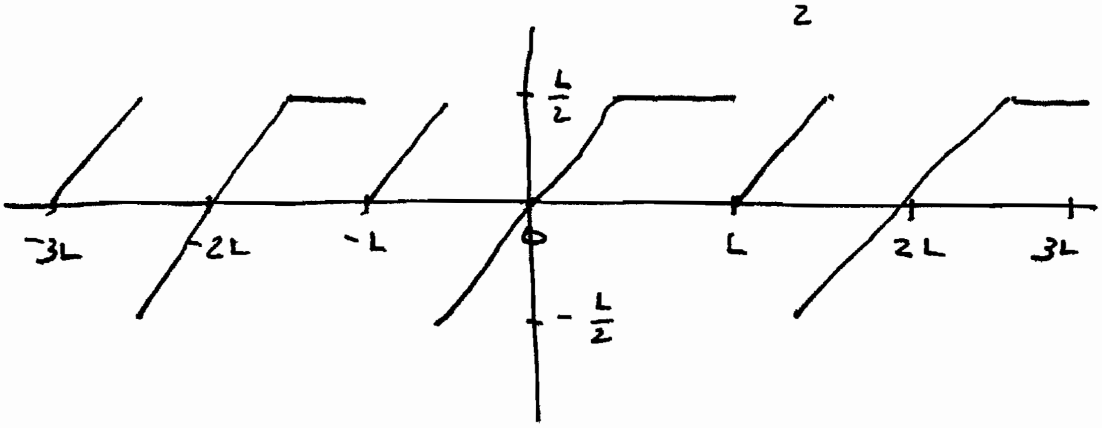
ME 201/MTH 281 FINAL EXAM SOLUTIONS DEC 19, 2008

(1) (a) f is piecewise smooth but has a discontinuity at $x = -\frac{L}{2}$.

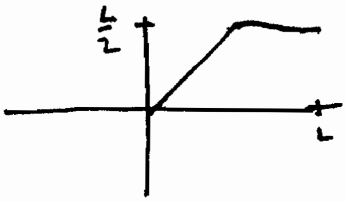


(b)

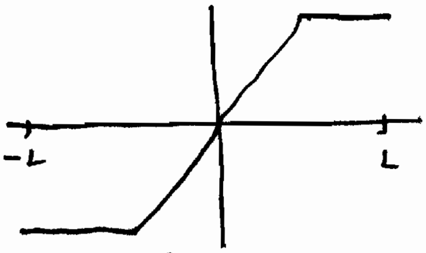


The extended function has discontinuities so the Fourier coefficients will drop off like $1/n$.

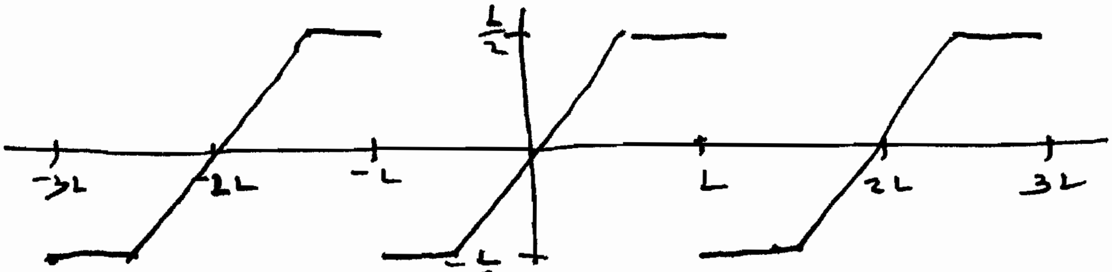
(c) For the sine series, we start with f on $[0,L]$:



. Then we make the odd extension to $[-L,L]$:

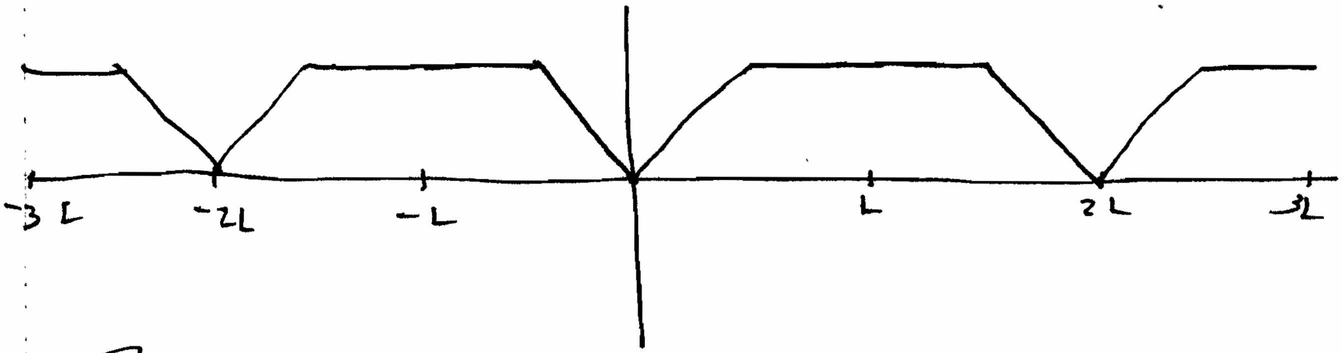


The function represented by the Fourier sine series is the periodic extension of this.



The extended function has discontinuities, so the convergence is like $1/n$.

(1) (continued) (d) For cosine series, we start with f on $[0, L]$, make an even extension to $[-L, 0]$, and then extend the function periodically.



The extended function is continuous but has discontinuities in slope, so the convergence will be like $1/n^2$.

(2) We try $T = F(x)G(y)$. We substitute into the equation and divide by FG:

$$\frac{1}{F}\frac{d^2F}{dx^2} = -\frac{1}{G}\frac{d^2G}{dy^2} = -\lambda.$$

Then $\dfrac{d^2F}{dx^2} + \lambda F = 0, \quad 0 < x < a, \quad F'(0) = 0, \quad F(a) = 0.$

The general solution is $F = A\cos\sqrt{\lambda}\,x + B\sin\sqrt{\lambda}\,x$.

$F'(0) = 0 = B\sqrt{\lambda} \Rightarrow B = 0$ (we check $\lambda = 0$ shortly). Then
$F(a) = 0 = A\cos\sqrt{\lambda}\,a$ so $\sqrt{\lambda}\,a = (n - \frac{1}{2})\pi$, $n = 1, 2, \ldots$
$\therefore \quad \lambda_n = (n - \frac{1}{2})^2\pi^2/a^2$, $F_n(x) = \cos[(n - \frac{1}{2})\pi x/a]$, $n = 1, 2, \ldots$

We check $\lambda = 0$. $\lambda = 0 \Rightarrow F''(x) = 0 \Rightarrow F(x) = Cx + D$.
$F'(0) = 0 \Rightarrow C = 0$. $F(a) = 0 \Rightarrow D = 0$ $\therefore$ trivial solution.

The y-equation is $\dfrac{d^2G}{dy^2} - \lambda G = 0$, so
$G = E\sinh\sqrt{\lambda}\,y + F\cosh\sqrt{\lambda}\,y$. We can impose the homogeneous BC at $y = 0$ on G: $\dfrac{dG}{dy}(0) = 0 \Rightarrow E = 0$.

(2) (continued) The separated solutions are then

$$\cosh\left[\left(n-\tfrac{1}{2}\right)\frac{\pi y}{a}\right]\cos\left[\left(n-\tfrac{1}{2}\right)\frac{\pi x}{a}\right].$$

We now superpose the solutions so that we can satisfy the inhomogeneous BC at $y=b$.

$$T(x,y)=\sum_{n=1}^{\infty} C_n \cosh\left[\left(n-\tfrac{1}{2}\right)\frac{\pi y}{a}\right]\cos\left[\left(n-\tfrac{1}{2}\right)\frac{\pi x}{a}\right]$$

Finally we impose the BC at $y=b$:

$$T(x,b)=T_0=\sum_{n=1}^{\infty} C_n \cosh\left[\left(n-\tfrac{1}{2}\right)\frac{\pi b}{a}\right]\cos\left[\left(n-\tfrac{1}{2}\right)\frac{\pi x}{a}\right]$$

We use orthogonality to get

$$C_n = \frac{T_0\int_0^a \cos\left[\left(n-\tfrac{1}{2}\right)\frac{\pi x}{a}\right]dx}{\cosh\left[\left(n-\tfrac{1}{2}\right)\frac{\pi b}{a}\right]\int_0^a \cos^2\left[\left(n-\tfrac{1}{2}\right)\frac{\pi x}{a}\right]}$$

$$= \frac{2T_0(-1)^{n+1}}{\pi\left(n-\tfrac{1}{2}\right)\cosh\left[\left(n-\tfrac{1}{2}\right)\frac{\pi b}{a}\right]}$$

$$\text{so}\quad T(x,y) = T_0\sum_{n=1}^{\infty}\frac{2(-1)^{n+1}\cosh\left[\left(n-\tfrac{1}{2}\right)\frac{\pi y}{a}\right]}{\cosh\left[\left(n-\tfrac{1}{2}\right)\frac{\pi b}{a}\right]}\,\frac{\cos\left[\left(n-\tfrac{1}{2}\right)\frac{\pi x}{a}\right]}{\pi\left(n-\tfrac{1}{2}\right)}.$$

(3) (a) $\int_0^1 (1+x^2)\,y_n(x)\,y_m(x)\,dx = 0$ for $n\neq m$.

(b) If $\lambda=0$, then $\frac{d}{dx}\left[(1+x)\frac{dy}{dx}\right]=0 \Rightarrow \frac{dy}{dx}=\frac{C}{1+x} \Rightarrow$
$y = C\ln(1+x)+D$. We impose the BC: $y(1)=0=C\ln 2+D$
so $D=-C\ln 2$ and $y(x)=C\ln\left(\frac{1+x}{2}\right)$. So
$y(0)-\alpha y'(0)=0 \Rightarrow C\ln\left(\frac{1}{2}\right)-\alpha C=0,\quad \alpha=\ln\left(\frac{1}{2}\right)=-\ln 2.$

(3) (c) Multiply the equation by y.

$$y[(1+x)y']' + \lambda(1+x^2)y^2 = 0.$$

We integrate the first term by parts:

$$[(1+x)y'y]' - (1+x)(y')^2 + \lambda(1+x^2)y^2 = 0.$$

Integrate from 0 to 1.

$$\lambda \int_0^1 (1+x^2)y^2\,dx = \int_0^1 (1+x)(y')^2\,dx - [(1+x)y'y]_0^1$$

$$= \int_0^1 (1+x)(y')^2\,dx - 2y'(1)y(1) + y'(0)y(0)$$

$$= \int_0^1 (1+x)(y')^2\,dx + \alpha(y'(0))^2$$

so

$$\lambda = \frac{\int_0^1 (1+x)(y')^2\,dx + \alpha(y'(0))^2}{\int_0^1 (1+x^2)y^2\,dx} \geq 0.$$

We already know that $\lambda = 0$ is not an eigenvalue for $\alpha > 0$, hence $\lambda > 0$.

(4) We take the Fourier transform of the equation, letting $\tilde{\Phi}(k,t) = \mathcal{F}\{\Phi(x,t)\}$. Then

$$\mathcal{F}\left\{\frac{\partial^2 \Phi}{\partial t^2}\right\} = \mathcal{F}\left\{c^2 \frac{\partial^2 \Phi}{\partial x^2}\right\}$$

or

$$\frac{d^2 \tilde{\Phi}}{dt^2} = -k^2 c^2 \tilde{\Phi}.$$

Then $\tilde{\Phi} = A\cos kct + B\sin kct$. We impose the initial conditions: $\tilde{\Phi}(k,0) = \tilde{f}(k) = A$, $\frac{d\tilde{\Phi}}{dt}(k,0) = 0 = Bkc$, so $B = 0$ and $\tilde{\Phi}(k,t) = \tilde{f}(k)\cos kct$.

Then by inversion

$$\Phi(x,t) = \frac{1}{2\pi}\int_{-\infty}^{\infty} \tilde{f}(k)\cos(kct)\,e^{ikx}\,dk.$$

(Exercise: Show from here that $\Phi(x,t) = \frac{1}{2}[f(x-ct) + f(x+ct)]$.)

(5) We have $P_2(\cos\phi) = \frac{1}{2}(3\cos^2\phi - 1)$, so the boundary condition is $\frac{\partial u}{\partial r}(a,\phi) = 2C P_2(\cos\phi)$.

We conclude from that that P_2 is the only Legendre polynomial in the solution. The separated solutions outside a sphere which go to zero as $r \to \infty$ are $\frac{P_n(\cos\phi)}{r^{n+1}}$, so we try

$$u(r,\phi) = A\frac{P_2(\cos\phi)}{r^3}. \quad \text{Then}$$

$$\frac{\partial u}{\partial r}(a,\phi) = -\frac{3A P_2(\cos\phi)}{a^4} = 2C P_2(\cos\phi),$$

so $A = -\frac{2Ca^4}{3}$ and

$$u(r,\phi) = -\frac{2Ca}{3}\left(\frac{a}{r}\right)^3 P_2(\cos\phi)$$

(6) We substitute $u = \cos\omega t\, F(r)$ into the equation to get $\frac{1}{r}\frac{d}{dr}\left(\frac{dF}{dr}\right) + k^2 F = 0$, $k^2 = \frac{\omega^2}{c^2}$.

This is Bessel's equation of order zero with a parameter k. The only solution well-behaved at $r=0$ is

$$F = J_0(kr).$$

The boundary condition at $r=a$ requires $J_0(ka)=0$, so

$$k_n a = \alpha_n, \quad \alpha_n = n\text{th zero of } J_0.$$

Then $\omega_n = \frac{c}{a}\alpha_n$. The lowest frequency is for $n=1$: $\omega_1 = \frac{c}{a}\alpha_1$. The frequency of the second mode is $\omega_2 = \frac{c\alpha_2}{a}$. The second mode spatial amplitude is $J_0\left(\alpha_2 \frac{r}{a}\right)$. This vanishes at the boundary ($r=a$) and on the nodal circle, $\alpha_2 \frac{r}{a} = \alpha_1$, so

$$r_{node} = \frac{\alpha_1}{\alpha_2} a.$$