ME 201/MTH 281 FINAL EXAM

7:15 PM – 10:15 PM, DECEMBER 19, 2008

The exam covers all of the material of the course. You may use any books, notes, or references that you like, but you may not exchange material during the exam. Do all six problems. The value of each problem is shown, and the total possible is 100. **BE SURE TO EXPLAIN YOUR WORK!** Wrong calculations with no explanation will receive little partial credit. You may pick up a copy of the solutions after the exam. If you would like your graded exam and your course grade mailed to you, fill out a mailing label when you turn in your exam. Good luck!

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(1) (17 points) Consider the function *f*(*x*) defined below on [-*L*, *L*].

$$f(x) = \left\{ \begin{array}{c} x + L, \text{ for } -L \le x < -\frac{L}{2}, \\ x, \text{ for } -\frac{L}{2} \le x \le \frac{L}{2}, \\ \frac{L}{2}, \text{ for } \frac{L}{2} < x \le L. \end{array} \right\}$$

(a) Sketch *f*(*x*) on the interval [-*L*, *L*]. Does *f* have any discontinuities on this interval? Is *f* piecewise smooth on this interval?

(b) Sketch three periods of the function represented by the Fourier series of *f*, with the base interval of the series being [-*L*, *L*]. How fast will the Fourier coefficients drop off with *n*?

(c) Sketch three periods of the function represented by the Fourier sine series of *f* on [0,*L*]. How fast do the Fourier coefficients drop off with *n*?

(d) Sketch three periods of the function represented by the Fourier cosine series of *f* on [0, *L*]. How fast do the Fourier coefficients drop off with *n*?

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(2) (17 points) Solve the boundary value problem given below for $T(x,y)$ in a rectangle.

$$\nabla^2 T = 0 \ , 0 < x < a, \ 0 < y < b,$$

$$\frac{\partial T}{\partial x}(0,y) = 0, \ T(a,y) = 0, \ \frac{\partial T}{\partial y}(x,0) = 0, \text{ and } \mathrm{T}(x,b) = T_0,$$

where T_0 is a constant.

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(3) (16 points) Consider the regular Sturm-Liouville problem given below for $y(x)$ on [0, 1].

$$\frac{d}{dx}\left[(1+x)\frac{dy}{dx}\right]+\lambda(1+x^2)y=0,\ 0<x<1, \text{ with } y(0)-\alpha y'(0)=0,\ y(1)=0.$$

(a) State the orthogonality condition satisfied by eigenfunctions belonging to different eigenvalues.

(b) For what values of α, if any, is $\lambda = 0$ an eigenvalue?

(c) Show that if $\alpha > 0$, all of the eigenvalues are positive.

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(4) (16 points) Use the Fourier transform to solve the initial-value problem for the wave equation, as specified below. You may assume that the initial function $f(x)$ has a Fourier transform $\tilde{f}(k)$. Express your answer in terms of a Fourier inversion integral. Here the wave speed c is a positive constant.

$$\frac{\partial^2\Phi}{\partial t^2}=c^2\frac{\partial^2\Phi}{\partial x^2},\ -\infty<x<\infty,\quad t>0, \text{ with } \Phi(x,0)=f(x),\ \frac{\partial\Phi}{\partial t}(x,0)=0.$$

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(5) (17 points) Solve the boundary value problem given below for the potential $\Psi(r,\phi)$ outside a sphere of radius a. Here r is the spherical radial coordinate and ϕ is the spherical polar angle. There is no dependence on the cylindrical angle θ because the boundary condition is independent of θ. The quantity c in the boundary condition is a positive constant.

$$\nabla^2\Psi=0, \text{ for } a<r<\infty \text{ and } 0\le\phi\le\pi,$$

$$\text{with } \Psi\to 0 \text{ as } r\to\infty \text{ and } \frac{\partial\Psi}{\partial r}(a,\phi)=c(3\cos^2(\phi)-1).$$

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(6) (17 points) Consider the free vibrations of a circular membrane of radius a. The membrane is clamped at the outer edge. Suppose further that the modes under consideration are radially symmetric, so that the motions depend only on the cylindrical radial coordinate r and the time t. If $\Psi(r,t)$ is the vertical displacement of the membrane, then Ψ satisfies the equation and boundary condition given below.

$$\frac{\partial^2\Psi}{\partial t^2}=C^2\frac{1}{r}\frac{\partial}{\partial r}\left(r\frac{\partial\Psi}{\partial r}\right),\ 0\le r<a,\ t>0, \text{ with } \Psi(a,t)=0.$$

Here the wave speed C is a positive constant. Look for standing vibrational modes of the form

$$\Psi(r,t)=\cos(\omega t)F(r).$$

Find the vibration frequencies for the first two modes. Your answer will be in terms of C, a, and zeros of a Bessel function. A nodal curve for a given mode is a curve on which the displacement is zero for that mode. The second mode has one nodal curve which is a circle. Find the radius of this nodal circle. Your answer will be in terms of a and two zeros of a Bessel function.
