ME 201/MTH 281 EXAM #2 SOLUTIONS NOV 13, 2008

(1) We look for separated solutions of the form $F(x)G(y)$. Because of the homogeneous boundary conditions at $x=0$ and $x=a$, we expect oscillatory behavior in x. We substitute the separated form into the equation to get

$$\frac{F''(x)}{F(x)} = -\frac{G''(y)}{G(y)} = -\lambda .$$

Then $F''(x) + \lambda F(x) = 0$, with $F(0)=0$ and $F(a)=0$. We have solved this problem several times. The result is $\lambda_n = n^2\pi^2/a^2$, $F_n(x) = \sin\left(\frac{n\pi x}{a}\right)$, $n=1,2,3\ldots$

The equation for G_n is

$$G_n''(y) - \lambda_n G_n(y) = 0,$$

so $G_n = A_n \cosh\left(\frac{n\pi y}{a}\right) + B_n \sinh\left(\frac{n\pi y}{a}\right)$.

We may impose the homogeneous boundary condition at $y=0$ on G_n: $\left.\frac{dG_n}{dy}\right|_{y=0} = 0 = B_n \frac{n\pi}{a} \Rightarrow B_n = 0$.

Then the separated solutions which satisfy the three homogeneous boundary conditions are

$$\cosh\left(\frac{n\pi y}{a}\right)\sin\left(\frac{n\pi x}{a}\right), \quad n=1,2,3\ldots$$

In order to satisfy the remaining inhomogeneous boundary condition, we superpose these solutions.

$$\Phi(x,y) = \sum_{n=1} C_n \cosh\left(\frac{n\pi y}{a}\right)\sin\left(\frac{n\pi x}{a}\right),$$

so

$$\Phi(x,b) = C_1\cosh\left(\frac{\pi b}{a}\right)\sin\left(\frac{\pi x}{a}\right) + C_2\cosh\left(\frac{2\pi b}{a}\right)\sin\left(\frac{2\pi x}{a}\right) + C_3\cosh\left(\frac{3\pi b}{a}\right)\sin\left(\frac{3\pi x}{a}\right) + \cdots = \alpha \sin\left(\frac{2\pi x}{a}\right).$$

We balance coefficients: $C_2\cosh\left(\frac{2\pi b}{a}\right) = \alpha$, $C_n = 0$ for $n \neq 2$. Then

$$\Phi(x,y) = \alpha \frac{\cosh\left(\frac{2\pi y}{a}\right)}{\cosh\left(\frac{2\pi b}{a}\right)} \sin\left(\frac{2\pi x}{a}\right).$$

(2) (a) We multiply the equation by F and integrate by parts.

$$F'' + \lambda F = 0$$
$$FF'' + \lambda F^2 = 0$$
$$(FF')' - (F')^2 + \lambda F^2 = 0$$

Now we integrate over $[0, L]$.

$$FF'\Big|_0^L - \int_0^L (F')^2 dx + \lambda \int_0^L F^2 dx = 0.$$

The first term vanishes because of the boundary conditions. Then

$$\lambda = \frac{\int_0^L (F')^2 dx}{\int_0^L F^2 dx} \geq 0.$$

To get $\lambda = 0$, we would need $F' \equiv 0 \Rightarrow F =$ constant, but $F(L) = 0 \Rightarrow$ constant $= 0$, hence $F \equiv 0$. That is, $\lambda = 0$ gives only the trivial solution. Thus $\lambda > 0$.

(b) The general solution of the differential equation is $F = A\cos\sqrt{\lambda}\,x + B\sin\sqrt{\lambda}\,x$. The condition $F'(0) = 0$ gives $B\sqrt{\lambda} = 0$. Because $\lambda > 0$, this gives $B = 0$. Then $F = \cos\sqrt{\lambda}\,x$ (or any multiple of that). Then

$$F(L) = 0 \Rightarrow \cos\sqrt{\lambda}\,L = 0 \Rightarrow \sqrt{\lambda}\,L = (n - \tfrac{1}{2})\pi,\ n = 1, 2, 3 \ldots$$

so

$$\lambda_n = \frac{(n-\frac{1}{2})^2\pi^2}{L^2},\quad F_n(x) = \cos\left[\frac{(n-\frac{1}{2})\pi x}{L}\right],\ n = 1, 2, 3 \ldots$$

(c) $1 = \sum_{n=1}^{\infty} C_n \cos\left[\frac{(n-\frac{1}{2})\pi x}{L}\right]$. We use orthogonality

to get
$$C_n = \frac{\int_0^L 1 \cdot \cos\left[\frac{(n-\frac{1}{2})\pi x}{L}\right] dx}{\int_0^L \left\{\cos\left[\frac{(n-\frac{1}{2})\pi x}{L}\right]\right\}^2 dx} = \frac{\sin\left[(n-\frac{1}{2})\pi\right]\frac{L}{\pi(n-\frac{1}{2})}}{\frac{L}{2}}$$

$$= \frac{2}{\pi(n-\frac{1}{2})}\sin\left[(n-\tfrac{1}{2})\pi\right] = \frac{2(-1)^{n+1}}{\pi(n-\frac{1}{2})}.$$

(3) The functions $\sin(n\pi x/L)$ are appropriate because they are the eigenfunctions of $\partial^2/\partial x^2$ which satisfy the same homogeneous boundary conditions as T. We substitute the expansion into the PDE:

$$\frac{\partial}{\partial t}\sum C_n(t)\sin\left(\frac{n\pi x}{L}\right) = D\frac{\partial^2}{\partial x^2}\sum C_n(t)\sin\left(\frac{n\pi x}{L}\right)$$

$$+\, \gamma e^{-\alpha t}\sin\left(\frac{2\pi x}{L}\right)$$

so
$$\sum\left(\frac{dC_n}{dt} + \frac{n^2\pi^2}{L^2}D\,C_n\right)\sin\left(\frac{n\pi x}{L}\right) = \gamma e^{-\alpha t}\sin\left(\frac{2\pi x}{L}\right).$$

We balance coefficients:

$$\frac{dC_n}{dt} + \frac{n^2\pi^2}{L^2}D\,C_n = 0 \quad \text{for } n \neq 2 \qquad (1)$$

$$\frac{dC_2}{dt} + \frac{4\pi^2}{L^2}D\,C_2 = \gamma e^{-\alpha t} \qquad (2)$$

The solution of (1) is $C_n(t) = C_n(0)\, e^{-\frac{n^2\pi^2}{L^2}Dt}$.

Using the hint, we find the solution of (2) to be

$$C_2(t) = A e^{-\frac{4\pi^2 Dt}{L^2}} + \frac{\gamma}{\frac{4\pi^2 D}{L^2} - \alpha}\, e^{-\alpha t}.$$

The initial condition is

$$T(x,0) = 0 = \sum C_n(0)\sin\left(\frac{n\pi x}{L}\right).$$

Balancing coefficients we get $C_n(0) = 0$.

$\therefore$ $C_n(t) \equiv 0$ for $n \neq 2$. For C_2, we have

$$C_2(0) = 0 = A + \frac{\gamma}{\frac{4\pi^2 D}{L^2} - \alpha}$$

so $A = -\dfrac{\gamma}{\frac{4\pi^2 D}{L^2} - \alpha}$ and

$$T(x,t) = \frac{\gamma}{\frac{4\pi^2 D}{L^2} - \alpha}\left(e^{-\alpha t} - e^{-\frac{4\pi^2 Dt}{L^2}}\right)\sin\left(\frac{2\pi x}{L}\right)$$