ME201/MTH281 EXAM #2

THURSDAY NOVEMBER 13, 2008, 2:00 – 3:15 and 3:25 - 4:40 PM

This exam covers homework assignments 5 through 8, and the following sections of the class notes: sections 3.2 through 3.6 of Chapter 3, Chapters 4 and 5, and sections 6.1 and 6.2 of Chapter 6. You may use any books, notes or reference material that you like, but you may not exchange reference material with anyone else. If you need additional information to work a problem, ask me for it, and, if it is appropriate, I will put it on the board. Do all three problems. The value of each is shown, and the total possible is 100. **BE SURE TO EXPLAIN YOUR WORK!** Wrong calculations with no explanation will receive very little partial credit. Solutions will be posted on the web when both exams are over. Graded exams will be returned in class on Monday. Good luck!

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(1) (35 points) Solve the boundary value problem given below for the Laplace equation in a rectangle. The quantity α appearing in the boundary condition at $y = b$ is a given positive constant.

$$\frac{\partial^2\Phi}{\partial x^2} + \frac{\partial^2\Phi}{\partial y^2} = 0 \quad , \quad 0 < x < a \ \text{ and } \ 0 < y < b \quad ,$$

$$\text{with } \ \Phi(0,\mathrm{y}) = 0 \ , \Phi(a,y) = 0 \ , \frac{\partial\Phi}{\partial y}(x,0) = 0 \ ,$$

$$\text{and } \Phi(x,b) = \alpha \sin\left(\frac{2\pi x}{a}\right).$$

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(2) (30 points) Consider the regular Sturm-Liouville system given below.

$$\frac{d^2F}{dx^2} + \lambda F = 0 \ , 0 < x < L \ , \ \ \frac{dF}{dx}(0) = 0 \ , F(L){=}0 \ .$$

(a) Show directly from the equation and boundary conditions that the eigenvalues are all positive.

(b) Find the eigenvalues λ_n and the eigenfunctions $F_n(x)$.

(c) Expand the function $f(x) = 1$ on $0 < x < L$ in a series of these eigenfunctions. (Hint: a useful integral is $\int_0^L \left\{\cos\left[\frac{(n-\frac{1}{2})\pi x}{L}\right]\right\}^2 dx = \frac{L}{2}$ when n is an integer.)

(OVER)

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(3) (35 points) Consider the initial value problem given below for heat conduction in a slab with a source.

$$\frac{\partial T}{\partial t} = D\frac{\partial^2 T}{\partial x^2} + \gamma e^{-\alpha t}\sin\left(\frac{2\pi x}{L}\right),\ 0 < x < L,\ t > 0,$$

where D, γ and α are positive constants, and where $\alpha \neq \frac{4\pi^2}{L^2}D$. The initial condition is $T(x,0) = 0$, and the boundary conditions are $T(0,t) = 0$ and $T(L,t) = 0$. Solve this problem by an eigenfunction expansion of the form

$$T(x,t) = \sum_{n=1}^{\infty} C_n(t)\sin\left(\frac{n\pi x}{L}\right).$$

Explain why the functions $\sin(n\pi x / L)$ are appropriate eigenfunctions for this problem.

(Hint: the solution of $\frac{dy}{dt} + by = ce^{-\alpha t}$, where b, c and α are constants, is

$$y(t) = Ae^{-bt} + \frac{c}{b-\alpha}e^{-\alpha t}.$$

where A is an arbitrary constant.)
