ME 201/MTH 281 EXAM #1 SOLUTIONS OCT 9, 2008

(1) We try $T = F(x)G(t)$. We substitute this form into the equation and then divide by FG to get

$$\frac{1}{G}\frac{dG}{dt} + \frac{U}{F}\frac{dF}{dx} = \frac{D}{F}\frac{d^2F}{dx^2} - \gamma e^{-\beta t}.$$

In this form there are x's and t's on both sides, so we transpose terms to get

$$\underbrace{\frac{1}{G}\frac{dG}{dt} + \gamma e^{-\beta t}}_{t \text{ only}} = \underbrace{\frac{D}{F}\frac{d^2F}{dx^2} - \frac{U}{F}\frac{dF}{dx}}_{x \text{ only}}$$

Now the separation has worked and each side is equal to the same constant λ. Then

$$\frac{1}{G}\frac{dG}{dt} + \gamma e^{-\beta t} = -\lambda \Rightarrow \frac{dG}{dt} + (\lambda + \gamma e^{-\beta t})G = 0$$

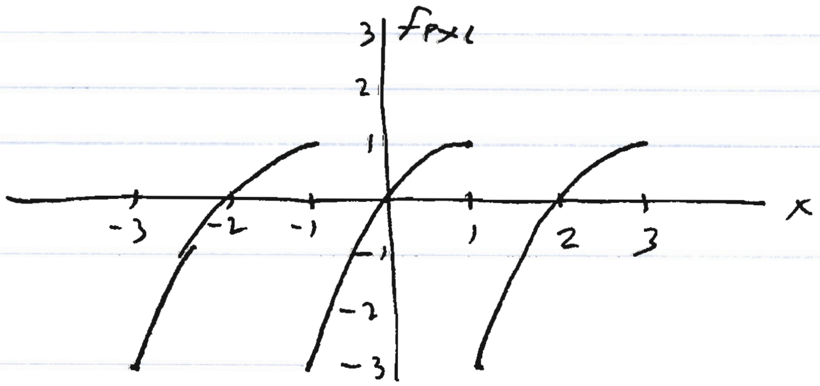
and

$$\frac{D}{F}\frac{d^2F}{dx^2} - \frac{U}{F}\frac{dF}{dx} = -\lambda \Rightarrow \frac{d^2F}{dx^2} - \frac{U}{D}\frac{dF}{dx} + \frac{\lambda}{D}F = 0.$$

The relevant boundary conditions on F follow from the given conditions on T at the end-points:

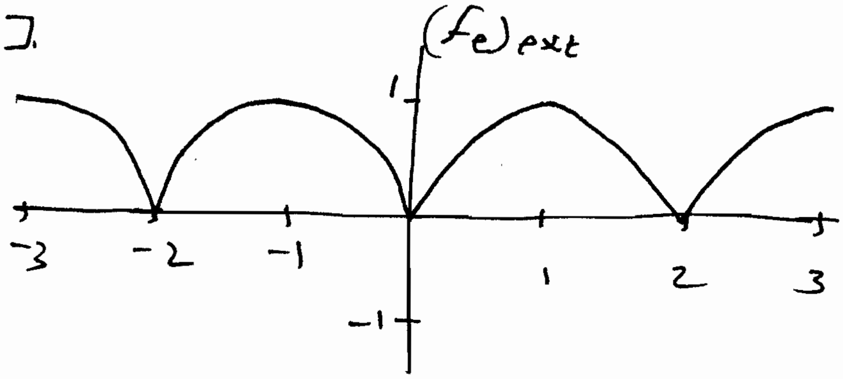
$$\frac{dF}{dx}(0) = 0, \quad F(L) = 0.$$

(2) (a) We sketch the function on $-1 \le x \le 1$ and then periodically extend it. f vanishes at 0 and 2, and has zero slope at 1, with $f(1) = 1$.



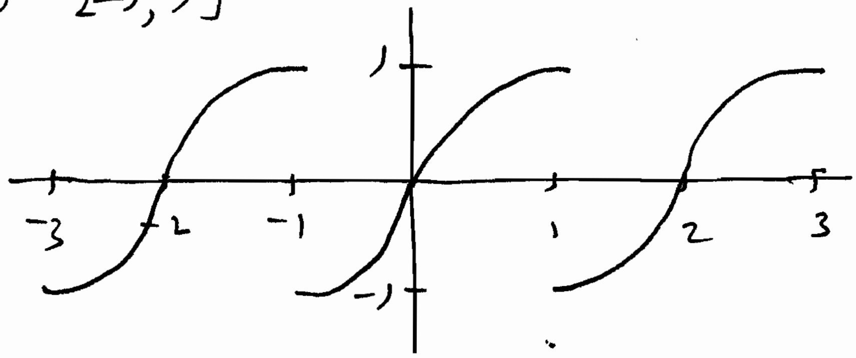
(2) (a) (continued) f_{ext} is piecewise smooth but it is not continuous, so the convergence will be like $\frac{1}{n}$.

(b) The extended function is the period 2 extension of the even extension of f from $[0,1]$ to $[-1,1]$.



Here the extended function is continuous but the slope is discontinuous at $0, \pm 2, \pm 4, \ldots$, so the convergence will be like $1/n^2$.

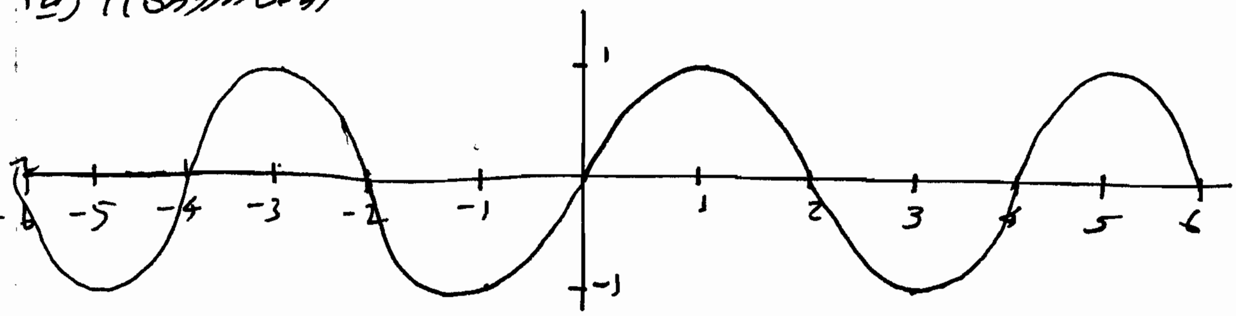
(c) The extended function is the period 2 extension of the odd extension of f from $[0,1]$ to $[-1,1]$



The extended function is discontinuous so the convergence will be like $1/n$.

(d) The extended function is the period 4 extension of the odd extension of f from $[0,2]$ to $[-2,2]$.

(2) (d) (continued)



$$f_o(x) = \begin{cases} 2x - x^2, & 0 \le x \le 1 \\ 2x + x^2, & -1 \le x \le 1 \end{cases}$$

Clearly f_{ext} is continuous. The derivative appears to be also. We check it.

$$f_o'(0+) = 2 = f_o'(0-)$$

$$f_o'(2-) = -2 = f_o'(-2+)$$

Because $f_o''(0+) = -2$ and $f_o''(0-) = 2$, the extended function has a discontinuous second derivative. The convergence will be like $1/n^3$.

(3) Because of the inhomogeneous boundary condition, we will need to split the solution into steady-state and transient parts: $T(x,t) = T_S(x) + \hat{T}(x,t)$.

For T_S we have $\frac{d^2 T_S}{dx^2} = 0 \Rightarrow T_S = Ax + B$.

The boundary conditions give $T_S(0) = 0 = B$, $T_S(L) = AL = T_0$ so $A = T_0/L$. Then $T_S = T_0 x/L$. The problem satisfied by $\hat{T}$ is

$$\frac{\partial \hat{T}}{\partial t} = D \frac{\partial^2 \hat{T}}{\partial x^2}, \quad 0 < x < L, \ t > 0, \text{ with } \hat{T}(0,t) = 0,$$

$\hat{T}(L,t) = 0$, and $\hat{T}(x,0) = T(x,0) - T_S(x) = T_0\left[\sin\left(\frac{\pi x}{L}\right) - \sin\left(\frac{3\pi x}{L}\right)\right]$.

(3) (continued) In class we showed that

$$\hat{T}(x,t) = \sum_{n=1}^{\infty} C_n e^{-n^2\pi^2 Dt/L^2} \sin(n\pi x/L),$$

where the coefficients C_n are determined by the initial condition

$$\hat{T}(x,0) = \sum_{n=1}^{\infty} C_n \sin(n\pi x/L) = T_0\left[\sin\left(\frac{\pi x}{L}\right) - \sin\left(\frac{3\pi x}{L}\right)\right]$$

By comparing coefficients, we get $C_1 = T_0$, $C_3 = -T_0$, all other $C_n = 0$. The final solution is

$$T(x,t) = T_S(x) + \hat{T}(x,t) = T_0 \frac{x}{L} + T_0\left[e^{-\pi^2 Dt/L^2} \sin(\pi x/L) - e^{-9\pi^2 Dt/L^2} \sin(3\pi x/L)\right]$$

(4) Diffusion times vary like the square of the thickness. Thus the protection time for the 0.5 m wall should be four times as long as that for the 0.25 m wall. Fire him (or at least make him take ME201/MTH 281 over).