ME 201/MTH 281 ASSIGNMENT # 9 SOLUTIONS NOV 20, 2008

(1) Let $f(x) = e^{-x^2}$. Then we know that $\mathcal{F}\{f\} = \sqrt{\pi}\, e^{-k^2/4}$. We have $f'(x) = -2x\, e^{-x^2}$ and $f''(x) = -2e^{-x^2} + 4x^2 e^{-x^2}$, so

$$x^2 e^{-x^2} = \frac{1}{2} f + \frac{1}{4} f''.$$

Then

$$\begin{aligned}\mathcal{F}\{x^2 e^{-x^2}\} &= \frac{1}{2}\mathcal{F}\{f\} + \frac{1}{4}\mathcal{F}\{f''\} \\ &= \frac{1}{2}\mathcal{F}\{f\} - \frac{k^2}{4}\mathcal{F}\{f\} \\ &= \frac{1}{4}(2-k^2)\mathcal{F}\{f\} = \frac{1}{4}(2-k^2)\sqrt{\pi}\, e^{-k^2/4}.\end{aligned}$$

(2) (a) By the chain rule $\frac{\partial}{\partial x} = \frac{\partial \hat{x}}{\partial x}\frac{\partial}{\partial \hat{x}} = \frac{1}{b}\frac{\partial}{\partial \hat{x}}$, so $\frac{\partial^2}{\partial x^2} = \frac{1}{b^2}\frac{\partial^2}{\partial \hat{x}^2}$. Similarly $\frac{\partial^2}{\partial y^2} = \frac{1}{b^2}\frac{\partial^2}{\partial \hat{y}^2}$. Then $\nabla^2\Phi = 0$ becomes $\frac{1}{b^2}\hat{\nabla}^2 \Phi_0 \hat{\Phi} = 0$ or $\hat{\nabla}^2\hat{\Phi} = 0$.

The domain is $0 < \hat{y} < 1$, $-\infty < \hat{x} < \infty$. The boundary condition at $\hat{y} = 0$ is $\hat{\Phi}(\hat{x}, 0)$, and the condition at $\hat{y} = 1$ is

$$\hat{\Phi}(\hat{x}, 1) = \frac{\Phi(x, b)}{\Phi_0} = \left(\frac{x}{b}\right)^2 e^{-\left(\frac{x}{b}\right)^2} = \hat{x}^2 e^{-\hat{x}^2}.$$

Summary:

$$\frac{\partial^2\hat{\Phi}}{\partial \hat{x}^2} + \frac{\partial^2\hat{\Phi}}{\partial \hat{y}^2} = 0, \quad 0 < \hat{y} < 1, \; -\infty < \hat{x} < \infty$$

$$\hat{\Phi} \to 0 \text{ as } \hat{x} \to \pm\infty$$

$$\hat{\Phi}(\hat{x}, 0) = 0, \quad \hat{\Phi}(\hat{x}, 1) = \hat{x}^2 e^{-\hat{x}^2}.$$

We drop the hats from here on.

(b) We transform the equation and use the derivative rule to get $\frac{d^2\tilde{\Phi}}{dy^2} - k^2\tilde{\Phi} = 0$,

where $\tilde{\Phi}(k, y) = \int_{-\infty}^{\infty} e^{-ikx}\,\Phi(x, y)\,dx$.

At $y = 0$, $\tilde{\Phi} = \mathcal{F}\{\Phi(x, 0)\} = \mathcal{F}\{0\} = 0$.

(2) (b) (continued) At $y=1$, $\tilde{\Phi} = \mathcal{F}\{\Phi(x,1)\} = \mathcal{F}\{x^2 e^{-x^2}\}$

$$= \frac{\sqrt{\pi}}{4}(2-k^2)e^{-k^2/4}.$$

The general solution of the equation for $\tilde{\Phi}$ is

$$\tilde{\Phi}(k,y) = A\cosh ky + B\sinh ky.$$

$y=0$: $\tilde{\Phi} = 0 = A \Rightarrow A=0$

$y=1$: $\tilde{\Phi} = \frac{\sqrt{\pi}}{4}(2-k^2)e^{-k^2/4} = B\sinh k$

$$\Rightarrow B = \frac{\sqrt{\pi}}{4}\frac{(2-k^2)e^{-k^2/4}}{\sinh k},$$

so $$\tilde{\Phi}(k,y) = \frac{\sqrt{\pi}}{4}(2-k^2)e^{-k^2/4}\frac{\sinh ky}{\sinh k}.$$

The Fourier inversion is

$$\Phi(x,y) = \frac{1}{2\pi}\int_{-\infty}^{\infty}\tilde{\Phi}(k,y)e^{ikx}\,dk = \frac{1}{8\sqrt{\pi}}\int_{-\infty}^{\infty}(2-k^2)e^{-\frac{k^2}{4}}\frac{\sinh ky}{\sinh k}e^{ikx}\,dk$$

(c) $$\Phi(0,y) = \frac{1}{8\sqrt{\pi}}\int_{-\infty}^{\infty}(2-k^2)e^{-\frac{k^2}{4}}\frac{\sinh ky}{\sinh k}\,dk.$$

The integrand is an even function of k, so we can write this as

$$\Phi(0,y) = \frac{1}{4\sqrt{\pi}}\int_{0}^{\infty}(2-k^2)e^{-\frac{k^2}{4}}\frac{\sinh ky}{\sinh k}\,dk.$$

(d) See Mathematica notebook.

(3) We introduce $\psi(x,t) = T/T_0$. Then ψ is a solution of $\frac{\partial\psi}{\partial t} = D\frac{\partial^2\psi}{\partial x^2}$, $-\infty<x<\infty$, $t>0$, with

$\psi(x,0) = 1$ for $x<0$ and $\psi(x,0)=0$ for $x>0$. ψ is a dimensionless function which depends on x, t and D. Because ψ is dimensionless, it can

(3) (continued) depend on these quantities only in a dimensionless combination. As we showed in class there is one such combination:

$$\eta = \frac{x}{2\sqrt{Dt}}.$$

So $\psi = F(\eta)$. We substitute this into the partial differential equation and solve the resulting equation for F. This was done in class. The result is

$$F(\eta) = A\int_0^{\eta} e^{-\eta'^2}\,d\eta' + B,$$

where A and B are constants to be determined.

As $t \to 0+$ for $x>0$, $\psi \to 0$, so $F \to 0$ as $\eta \to \infty$

$$0 = A\int_0^{\infty} e^{-\eta'^2}\,d\eta' + B = \frac{\sqrt{\pi}}{2}A + B.$$

As $t \to 0+$ for $x<0$, $\eta \to -\infty$ and we must have $F \to 1$. So

$$1 = A\int_0^{-\infty} e^{-\eta'^2}\,d\eta' + B = -\frac{\sqrt{\pi}}{2}A + B.$$

We solve these two equations to get $A = -\frac{1}{\sqrt{\pi}}$, $B = \frac{1}{2}$

Then

$$F(\eta) = -\frac{1}{\sqrt{\pi}}\int_0^{\eta} e^{-\eta'^2}\,d\eta' + \frac{1}{2}$$

$$= \frac{1}{2}\left(1 - \operatorname{erf}(\eta)\right) = \frac{1}{2}\operatorname{erfc}(\eta)$$

So $T(x,t) = \dfrac{T_0}{2}\operatorname{erfc}(\eta)$

(4) The radius R will depend on E, ρ, and t. For the ~~units~~ dimensions of these quantities we may use either a mass (M), time (T) and length (L) system, or a force (F), time (T) and length (L) system. We choose the first. The ~~un~~ dimensions of the relevant quantities are then

$$[R] = L, \quad [E] = M\frac{L^2}{T^2}, \quad [\rho] = \frac{M}{L^3}, \quad [t] = T.$$

We start by making R dimensionless. A little experimenting shows that

$$\frac{R}{\left(\frac{Et^2}{\rho}\right)^{1/5}}$$

is dimensionless. Therefore this quantity must depend on E, t, ρ only in a a dimensionless combination, but there are no dimensionless combinations of those three quantities. Therefore the ratio is a pure dimensionless constant C, and we have

$$R = C\left(\frac{Et^2}{\rho}\right)^{1/5}$$

So the fireball grows like $t^{2/5}$.

(5) (a) We clear fractions. $(1+x)y'' + y = 0$.

$$y = a_0 + a_1 x + a_2 x^2 + \cdots = \sum_0^\infty a_n x^n.$$

$$y'' = 2a_2 + 6a_3 x + (4)(3)a_4 x^2 + \cdots = \sum (n+1)(n+2)\, a_{n+2}\, x^n$$

$$xy'' = 2a_2 x + 6a_3 x^2 + (4)(3)a_4 x^3 + \cdots = \sum (n)(n+1)\, a_{n+1}\, x^n$$

(5) (a) (continued) We substitute these into the equation.

$$\sum (n+1)(n+2)a_{n+2}x^n + \sum n(n+1)a_{n+1}x^n + \sum a_n x^n = 0.$$

Balancing coefficients we get

$$(n+1)(n+2)a_{n+2} + n(n+1)a_{n+1} + a_n = 0,$$

so

$$a_{n+2} = -\frac{n}{n+2}a_{n+1} - \frac{a_n}{(n+1)(n+2)}.$$

From the initial conditions we get $a_0 = 1$, $a_1 = -1$. Then we use the above formula.

$$a_2 = -\frac{0}{2}a_1 - \frac{a_0}{2} = -\frac{a_0}{2} = -\frac{1}{2}$$

$$a_3 = -\frac{1}{3}a_2 - \frac{a_1}{(2)(3)} = -\frac{a_2}{3} - \frac{a_1}{6}$$

$$= \frac{1}{6} + \frac{1}{6} = \frac{1}{3}.$$

So

$$y(x) = 1 - x - \frac{1}{2}x^2 + \frac{1}{3}x^3 + \cdots.$$

(b) $y'' + 4y = 0$, $y(0) = 0$, $y'(0) = -2$.

We let

$$y(x) = a_0 + a_1 x + a_2 x^2 + \cdots = \sum_{n=0}^{\infty} a_n x^n$$

$$y'' = 2a_2 + (2)(3)a_3 x + (3)(4)a_4 x^2 + \cdots$$

$$= \sum_{n=0}^{\infty} (n+1)(n+2)a_{n+2}x^n$$

The equation is then

$$\sum (n+1)(n+2)a_{n+2}x^n + 4\sum a_n x^n = 0.$$

We balance the coefficients of x^n to get

$$(n+1)(n+2)a_{n+2} + 4a_n = 0$$

(5) (b) (continued)

$$\text{so} \quad a_{n+2} = -\frac{4a_n}{(n+1)(n+2)}$$

We have $a_0 = 0$, $a_1 = -2$ from the initial conditions. It is clear that all even coefficients will vanish. We calculate the odd coefficients.

$$a_3 = -\frac{4a_1}{(2)(3)}$$

$$a_5 = -\frac{4a_3}{(4)(5)} = \frac{4^2 a_1}{5!}$$

$$a_7 = -\frac{4a_5}{(6)(7)} = -\frac{4^3 a_1}{7!}$$

$$\vdots$$

$$a_{2n+1} = \frac{(-1)^n 4^n a_1}{(2n+1)!}$$

$$\text{so} \quad y(x) = \frac{a_1}{2} \sum_{n=0}^{\infty} \frac{(-1)^n x^{2n+1} 2^{2n+1}}{(2n+1)!}$$

$$= \frac{a_1}{2} \sum_{n=0}^{\infty} \frac{(-1)^n (2x)^{2n+1}}{(2n+1)!} = \frac{a_1}{2} \sin(2x)$$

$$= -\sin(2x)$$

a solution which is easy to obtain by using the basic techniques of MTH 163 or 165.

(c) We clear fractions to get

$$x^2 y'' - 2xy' + 2y = 0.$$

(5) (c) (continued) The easiest way to carry out an expansion about $x=1$ is to introduce $\eta = x-1$ and expand in powers of η. By the chain rule, $\frac{d}{dx} = \frac{d\eta}{dx}\frac{d}{d\eta} = \frac{d}{d\eta}$.

So

$$(\eta+1)^2 \frac{d^2y}{d\eta^2} - 2(\eta+1)\frac{dy}{d\eta} + 2y = 0.$$

There will be six terms in the equation:

$$(\eta^2+2\eta+1)\frac{d^2y}{d\eta^2} - 2(\eta+1)\frac{dy}{d\eta} + 2y = 0.$$

$$y = a_0 + a_1\eta + a_2\eta^2 + \cdots = \sum a_n \eta^n$$

$$\frac{dy}{d\eta} = a_1 + 2a_2\eta + 3a_3\eta^2 + \cdots = \sum (n+1)a_{n+1}\eta^n$$

$$\eta\frac{dy}{d\eta} = a_1\eta + 2a_2\eta^2 + 3a_3\eta^3 + \cdots = \sum (n) a_n \eta^n$$

$$\frac{d^2y}{d\eta^2} = 2a_2 + (2)(3)a_3\eta + (3)(4)a_4\eta^2 \cdots$$
$$= \sum (n+1)(n+2)a_{n+2}\eta^n$$

$$\eta\frac{d^2y}{d\eta^2} = 2\eta a_2 + (2)(3)a_3\eta^2 + \cdots = \sum (n)(n+1)a_{n+1}\eta^n$$

$$\eta^2\frac{d^2y}{d\eta^2} = 2a_2\eta^2 + (2)(3)a_3\eta^3 + \cdots = \sum (n-1)(n)a_n\eta^n$$

We substitute all of these in to the equation.

$$\sum (n-1)(n)a_n\eta^n + 2\sum n(n+1)a_{n+1}\eta^n + \sum (n+1)(n+2)a_{n+2}\eta^n$$
$$-2\sum n a_n\eta^n - 2\sum (n+1)a_{n+1}\eta^n + 2\sum a_n\eta^n = 0.$$

(5) (c) (continued) We balance the coefficients of η^n:

$$(n+1)(n+2)\,a_{n+2} + \left[2n(n+1) - 2(n+1)\right] a_{n+1}$$

$$+ \left[(n-1)(n) - 2n + 2\right] a_n = 0$$

$$(n+1)(n+2)\,a_{n+2} + 2(n-1)(n+1)\,a_{n+1}$$

$$+ (n-1)(n-2)\,a_n = 0$$

So $$a_{n+2} = -\frac{2(n-1)}{n+2}\,a_{n+1} - \frac{(n-1)(n-2)}{(n+1)(n+2)}\,a_n$$

We get $a_2 = a_1 - a_0$.

$$a_3 = 0\cdot a_2 - 0\cdot a_1 = 0$$

$$a_4 = -\tfrac{1}{2}\,a_3 - 0\cdot a_1 = 0.$$

Because two consecutive a's vanish, it is clear from the three term recurrence formula that all higher coefficients vanish. Thus the series truncates, and the only non-zero coefficients are

$$a_0 = 1, \quad a_1 = 2 \quad \text{(from initial conditions)}$$

and $a_2 = a_1 - a_0 = 1$

Then $y = 1 + 2\eta + \eta^2 = (\eta+1)^2 = x^2$.
It is much easier to find this solution by treating the original equation as an equi-dimensional equation in x.

ME 201/ MTH 281
2008
Assignment #9

■ Problem 2 d

We use the command NIntegrate (numerical integration) to define the potential on the y-axis, using the Fourier inversion formula as the basis for the definition. We call the function f[y].

```
f[y_] :=
 (1 / (4 Sqrt[π])) NIntegrate[(2 - k^2) Exp[-k^2 / 4] Sinh[k y]/Sinh[k], {k, 0, ∞}]
```

$$f[y\_] := \left(1 \Big/ \left(4\sqrt{\pi}\right)\right) \text{NIntegrate}\left[\left(2-k^2\right) \text{Exp}\left[-k^2/4\right] \frac{\text{Sinh}[k\,y]}{\text{Sinh}[k]}, \{k, 0, \infty\}\right]$$

The potential vanishes on both boundaries at y = 0, and we can use this to get a partial check of our formula.

```
f[0]
```

— NIntegrate::ncvb :
NIntegrate failed to converge to prescribed accuracy after 9 recursive bisections in k near {k} = {0.501431}. NIntegrate obtained 0.` and 0.` for the integral and error estimates. ≫

```
0.
```

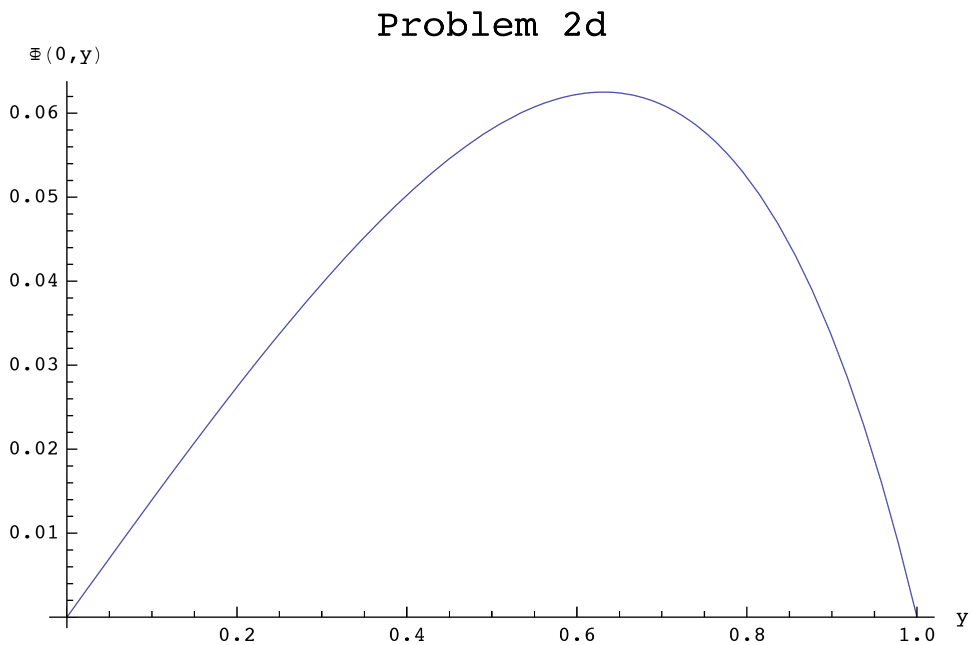
```
f[1]
```

— NIntegrate::ncvb :
NIntegrate failed to converge to prescribed accuracy after 9 recursive bisections in k near {k} = {3.45187}. NIntegrate obtained -3.26128×10^{-16} and 4.2525209040648975`*^-16 for the integral and error estimates. ≫

$0.\times 10^{-17}$

Mathematica fussed a bit about the convergence but still gave the correct zero answers. Now we use this function to construct a plot of the potential on the y-axis versus y.

```
Plot[f[y], {y, 0, 1},
 AxesLabel → {"y", "Φ(0,y)"}, PlotLabel → "Problem 2d"]
```



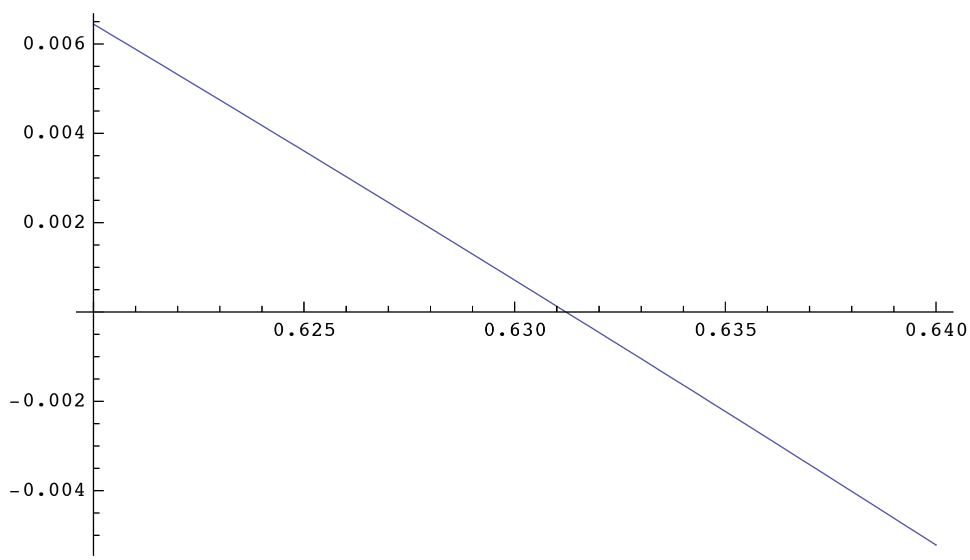
We see that the maximum is about 0.6 at y ≈ 0.63. We can use FindRoot to get a more accurate estimate. We differentiate the expression for the potential with respect to y. We call the derivative g[y].

```
g[y_] :=
 (1/(4 √π)) NIntegrate[k (2 - k^2) Exp[-k^2/4] Cosh[k y]/Sinh[k], {k, 0, ∞}]
```

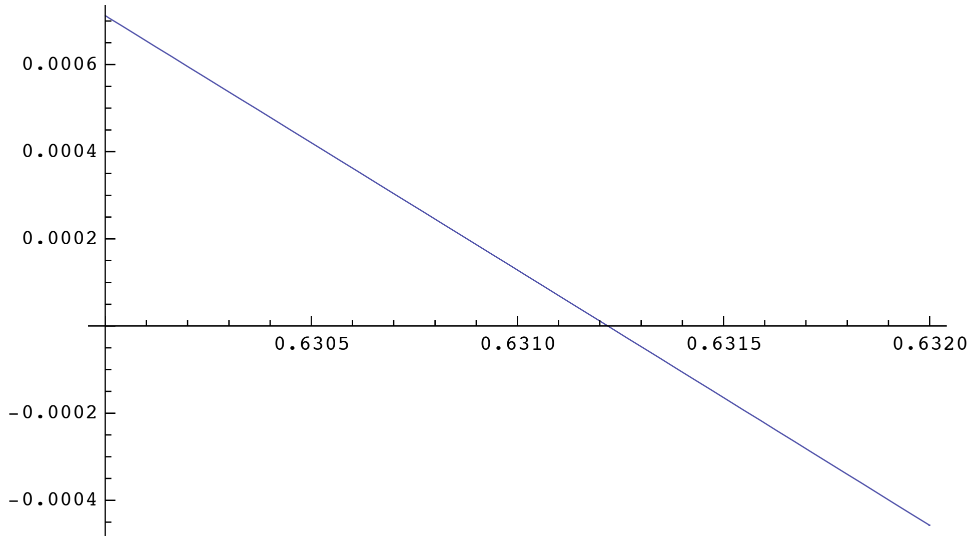
$$g[y\_] := \left(1 \Big/ \left(4\sqrt{\pi}\right)\right) \text{NIntegrate}\left[k\left(2-k^2\right) \text{Exp}\left[-k^2/4\right] \frac{\text{Cosh}[k\,y]}{\text{Sinh}[k]},\ \{k, 0, \infty\}\right]$$

Now we find graphically the location where the derivative vanishes.

```
Plot[g[y], {y, 0.62, 0.64}]
```



```
Plot[g[y], {y, 0.63, 0.632}]
```



We see that the zero is at y = 0.6312.

```
g[0.6312]
```

```
0.0000113708
```

```
f[0.6312]
```

```
0.0625135
```

This last number is our maximum value of the potential on the y-axis.