ME 201/MTH 281 ASSIGNMENT # 8 SOLUTIONS NOV 6, 2008

(1) (a) Following the hint we try $y = \sin kx \cos \omega t$. The equation gives

$$-\omega^2 = -\sigma k^4 \quad \text{or} \quad k = \left(\frac{\omega^2}{\sigma}\right)^{\frac{1}{4}}$$

We see that $y(0)=0$ and $y(L)=0$ provided $\sin kL = 0$, so $kL = n\pi$, $n=1,2,3\ldots$. The function $\sin(n\pi x/L)$ also has a zero second derivative at both $x=0$ and $x=L$. Hence our normal modes are

$$y_n = \cos \omega_n t \sin(n\pi x/L)$$

$$\text{where} \quad \omega_n = \sqrt{\sigma}\,\frac{n^2\pi^2}{L^2}.$$

Here is a systematic treatment which does not require an inspired guess. We try

$$y(x,t) = \cos\omega t\, F(x).$$

Substitution into the equation gives

$$-\omega^2 \cos\omega t\, F(x) = -\sigma \cos\omega t \frac{d^4F}{dx^4}$$

$$\text{so} \quad \frac{d^4F}{dx^4} - k^4 F = 0 \quad \text{where } k^4 = \frac{\omega^2}{\sigma}.$$

The boundary conditions are

$$F(0)=0, \quad F''(0)=0$$
$$F(L)=0, \quad F''(L)=0.$$

Before proceeding with general k, we check to see whether $k=0$ is possible. For $k=0$, $d^4F/dx^4=0 \Rightarrow F = A + Bx + Cx^2 + Dx^3$. $F(0)=0 \Rightarrow A=0$. $F''(0)=0 \Rightarrow C=0$. $F(L)=0 \Rightarrow B + DL^2 = 0$. $F''(L)=0 \Rightarrow 6DL = 0$, so $A=B=C=D=0$. Only the trivial solution.

(1) (a) (continued) For $k \neq 0$ we have

$$\frac{d^4F}{dx^4} - k^4 F = 0$$

so

$$\left(\frac{d^2}{dx^2} + k^2\right)\left(\frac{d^2}{dx^2} - k^2\right)F = 0$$

$$\left(\frac{d^2}{dx^2} - k^2\right)F = 0 \Rightarrow F = A\cosh kx + B\sinh kx$$

$$\left(\frac{d^2}{dx^2} + k^2\right)F = 0 \Rightarrow F = C\cos kx + D\sin kx.$$

The general solution is

$$F(x) = A\cosh kx + B\sinh kx + C\cos kx + D\sin kx.$$

We impose the BC's at $x=0$.

$$\left.\begin{array}{l} F(0) = 0 = A + C \\ F''(0) = 0 = k^2(A - C) \end{array}\right\} \Rightarrow A = C = 0.$$

We impose the BC's at $x = L$:

$$F(L) = 0 = B\sinh kL + D\sin kL$$

$$F''(L) = 0 = k^2(B\sinh kL - D\sin kL)$$

For a non-trivial solution the determinant must vanish.

$$\sinh kL\, B + \sin kL\, D = 0$$

$$\sinh kL\, B - \sin kL\, D = 0$$

determinant $= -2\sinh kL \sin kL$. $\sinh kL$ never vanishes for $k \neq 0$, so we must have

$$\sin kL = 0 \quad \Rightarrow B = 0$$

$\therefore$ $kL = n\pi$ and

$$F_n(x) = \sin\left(\frac{n\pi x}{L}\right), \quad \omega_n = \sqrt{\sigma}\,\frac{n^2\pi^2}{L^2}$$

where $n = 1, 2, 3 \ldots$

(1) (continued) (b) We are given that $\frac{\omega_1}{2\pi} = 2.33$ Hz.

Then $\frac{\omega_2}{\omega_1} = 4$, so $\frac{\omega_2}{2\pi} = 4(2.33) = 9.32$ Hz.

To find ρ, we use $\omega_1 = \sqrt{\sigma}\,\frac{\pi^2}{L^2}$, so

$$\frac{\omega_1}{2\pi} = 2.33 = \sqrt{\sigma}\,\frac{\pi}{2L^2}$$

so $$\sqrt{\sigma} = \frac{(2)(1\,m)^2(2.33\,s^{-1})}{\pi} = 1.483\,\frac{m^2}{s}$$

and $\sigma = 2.200\ m^4/s^2$. Then

$$\rho = \frac{EI}{\sigma A} \quad \text{where } E = 2.07\times 10^{11}\ N/m^2,$$

$$I = \frac{wb^3}{12} = \frac{(3\times 10^{-2})(10^{-3})^3}{12} = 0.25\times 10^{-11}\ m^4$$

and $A = (3\times 10^{-2})(10^{-3}) = 3\times 10^{-5}\ m^2$. So

$$\rho = \frac{(2.07\times 10^{11})(0.25\times 10^{-11})}{(2.200)(3\times 10^{-5})} = 7841\ kg/m^3$$

which is a typical density for steel.

(2) Following the procedure used in class for the acoustic normal modes, we try

$$\psi(x,y,t) = \cos\omega t\,\sin\left(\frac{m\pi x}{a}\right)\sin\left(\frac{n\pi y}{a}\right).$$

The spatial functions here meet the requirements that they are eigenfunctions of the Laplace operator and they satisfy the boundary conditions at the edge of the membrane. Without this convenient guess, we could still arrive at this form by trying

$$\psi(x,y,t) = \cos\omega t\ \Phi(x,y),$$

and separating variables in the resulting equation for Φ.

(2) (continued) ~~(e)~~ By substituting the expression $\psi = \cos\omega t \sin(m\pi x/a)\sin(n\pi y/a)$ into the equation we get

$$\omega_{mn}^2 = \frac{c^2\pi^2}{a^2}(m^2+n^2), \quad m,n = 1,2,3\ldots$$

(a) The fundamental corresponds to $m=n=1$, so

$$\omega_{fund}^2 = \frac{2c^2\pi^2}{a^2},$$

and $$\nu_{fund}^2 = \frac{c^2}{2a^2}, \quad \nu_{fund} = \frac{c}{\sqrt{2}a}.$$

Middle C has a frequency of 262 Hz, so an octave below is half this, or ~~131 Hz~~

$$\nu_0 = 131\text{ Hz}.$$

Then $$\frac{c}{\sqrt{2}a} = \nu_0, \quad \text{so } \sqrt{\frac{T}{\sigma}} = \sqrt{2}a\nu_0$$

hence
$$\begin{aligned} T &= 2\sigma a^2\nu_0^2 \\ &= (2)(0.1\text{ kg/m}^2)(0.5\text{ m})^2(131\text{ Hz})^2 \\ &= 858\text{ N/m}. \end{aligned}$$

(b) Let $\nu_{max} = 5000$ Hz. Then we want to know how many modes satisfy

$$\nu_{mn}^2 \le \nu_{max}^2$$

or $$\frac{c^2}{4a^2}(m^2+n^2) \le \nu_{max}^2$$

or $$(m^2+n^2) \le R^2, \quad R = \frac{2a\nu_{max}}{c}$$

This is the equation of a circle, of radius R, in lattice space. Each mode is associated with unit area.

(2) (b) (Continued). The area of the ~~[illegible]~~ part of the circle in the first quadrant is then the number of modes:

$$\frac{1}{4}\pi R^2 = \frac{1}{4}\pi\left(\frac{2a\,\nu_{max}}{c}\right)^2$$

$$= \frac{\pi a^2 \nu_{max}^2}{c^2} = \frac{\pi\, a^2 \nu_{max}^2 \sigma}{T}$$

$$= \frac{\pi\,(0.5\,m)^2 (5000)^2 (0.1\,kg/m^2)}{858\ N/m}$$

$$= 2290 \text{ modes}$$

(3) See Mathematica notebook.

(4) See Mathematica notebook

ME 201/MTH 281 2008 Assignment #8

Problem 3

Part (a)

We use *Mathematica* to evaluate the integrals for the Fourier transform. We first define the two pieces of f(x).

```
fl[x_] := Exp[a x]
```

```
fr[x_] := x Exp[-a x]
```

Now the Fourier transform.

```
kern[k_, x_] := Exp[-ⅈ k x]
```

```
ftrans[k_] = Integrate[fl[x] kern[k, x],
   {x, -∞, 0}, Assumptions → {k ∈ Reals, a > 0}] + Integrate[
   fr[x] kern[k, x], {x, 0, ∞}, Assumptions → {k ∈ Reals, a > 0}]
```

$$\frac{1}{a - i k} + \frac{1}{(a + i k)^2}$$

Part (b)

First we evaluate the integral.

```
Integrate[fl[x], {x, -∞, 0}, Assumptions → a > 0] +
 Integrate[fr[x], {x, 0, ∞}, Assumptions → a > 0]
```

$$\frac{1}{a^2} + \frac{1}{a}$$

Now we evaluate the Fourier transform at k = 0.

```
ftrans[0]
```

$$\frac{1}{a^2} + \frac{1}{a}$$

We have verified the equality.

Part (c)

Parseval's theorem says that $\int_{-\infty}^{\infty} |f(x)|^2 dx = \frac{1}{2\pi}\int_{-\infty}^{\infty} |\text{ftrans}(k)|^2 dk$. We start by evaluating the left-hand side.

```
Integrate[(Abs[fl[x]])^2, {x, -∞, 0}, Assumptions → a > 0] +
 Integrate[(Abs[fr[x]])^2, {x, 0, ∞}, Assumptions → a > 0]
```

$$\frac{1}{4\,a^3} + \frac{1}{2\,a}$$

Next we get the square of the absolute value of the Fourier transform. Then we will integrate it.

```
fsqr[k_] = ComplexExpand[ftrans[k] * Conjugate[ftrans[k]]]
```

$$\frac{a^4}{(a^2+k^2)^4} + \frac{2\,a^2\,k^2}{(a^2+k^2)^4} + \frac{k^4}{(a^2+k^2)^4} + \frac{2\,a^3}{(a^2+k^2)^3} - \frac{6\,a\,k^2}{(a^2+k^2)^3} + \frac{a^2}{(a^2+k^2)^2} + \frac{k^2}{(a^2+k^2)^2}$$

```
Simplify[(1 / (2 π)) Integrate[fsqr[k], {k, -∞, ∞}, Assumptions → a > 0]]
```

$$\frac{1+2\,a^2}{4\,a^3}$$

We get the same thing, thereby verifying Parseval's theorem in this case.

Problem 4

Part (a)

We are going to transform the entire differential equation, and we start by finding the Fourier transform of the right-hand side.

```
rs[x_] := x Exp[-Abs[x]]
```

```
rstrans[k_] = Simplify[
  Integrate[kern[k, x] rs[x], {x, -∞, ∞}, Assumptions → k ∈ Reals]]
```

$$-\frac{4\,i\,k}{(1+k^2)^2}$$

The Fourier transform of the left-hand side is obtained by using the derivative rule. The result is -(1 + k^2)

ytrans(k), so we may solve for ytrans:

```
ytrans[k_] = -rstrans[k] / (1 + k^2)
```

$$\frac{4\,i\,k}{\left(1+k^2\right)^3}$$

■ Part (b)

We use *Mathematica* to invert the Fourier transform.

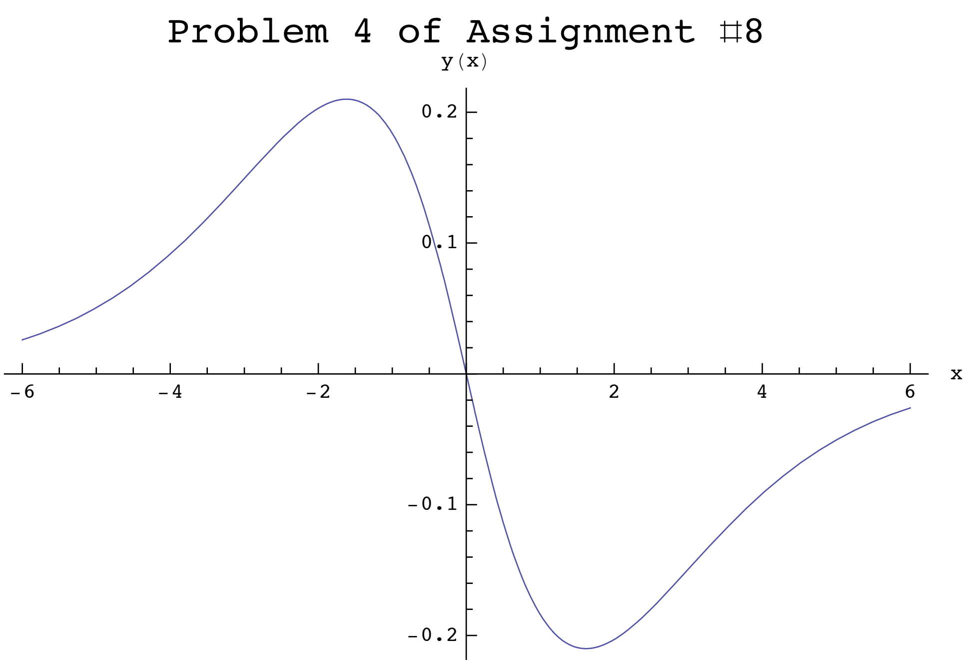
```
y[x_] = (1 / (2 π))
  Integrate[ytrans[k] Exp[i k x], {k, -∞, ∞}, Assumptions → x ∈ Reals]
```

$$-\frac{1}{4}\,e^{-\text{Abs}[x]}\,x\,(1+\text{Abs}[x])$$

■ Part (c)

We plot our solution from -6 to 6.

```
Plot[y[x], {x, -6, 6}, AxesLabel → {"x", "y(x)"},
 PlotLabel → "Problem 4 of Assignment #8"]
```



We estimate from the plot that the minimum and maximum values of y are -0.21 and 0.21. We can find a more accurate value by basic calculus techniques.

```
FindRoot[D[y[x], x] == 0, {x, 2}]
```

— FindRoot::nlnum :
The function value $\{-0.101501 + 0.135335\, \text{Abs}'[2.]\}$ is not a list of numbers with dimensions {1} at {x} = {2.}. ≫

```
FindRoot[∂x y[x] == 0, {x, 2}]
```

Mathematica is struggling with the Abs function, so we write the expression in simpler form, valid for x>0.

```
ysimp[x_] = -(1/4) x (1 + x) Exp[-x]
```

$$-\frac{1}{4} e^{-x} x (1 + x)$$

```
FindRoot[D[ysimp[x], x] == 0, {x, 2}]
```

$\{x \to 1.61803\}$

```
ysimp[x] /. %
```

-0.209991

So the range of the function is from -0.210 to 0.210, and our graphical estimate was right on.