ME 201/MTH 281 ASSIGNMENT # 6 SOLUTIONS OCT 24, 2008

(1) (a) We multiply the equation by ψ and integrate from $x=1$ to $x=2$. The integrand on the left-hand side becomes

$$\psi(x\psi')' = (x\psi\psi')' - x\psi'^2,$$

so $$\int_1^2 \psi(x\psi')'dx = x\psi\psi'\Big]_1^2 - \int_1^2 x\psi'^2 dx$$

The first term on the right is zero because $\psi(1)=\psi(2)=0$

The right-hand side is

$$-\lambda\int_1^2 \frac{\psi^2}{x}dx,$$

so we get

$$\lambda = \frac{\int_1^2 x\psi'^2 dx}{\int_1^2 \frac{\psi^2}{x}dx}.$$

These integrals are positive definite so $\lambda > 0$ unless $\psi'=0$ everywhere. If $\psi'=0$, then $\psi=$ constant and the boundary conditions tell us that the constant is zero, hence the solution is trivial.

$\therefore$ all eigenvalues are positive.

(b) We try $\psi = x^r$. Substituting this into the equation we get

$$r^2x^{r-1} = -\lambda x^{r-1} \Rightarrow r^2=-\lambda,\ r=\pm i\sqrt{\lambda}.$$

Then the solutions are

$$x^{i\sqrt{\lambda}} = \cos[\sqrt{\lambda}\ln x] + i\sin[\sqrt{\lambda}\ln x]$$

$$x^{-i\sqrt{\lambda}} = \cos[\sqrt{\lambda}\ln x] - i\sin[\sqrt{\lambda}\ln x]$$

We get a more convenient solution basis by taking

$$\psi_1 = \frac{1}{2}(x^{i\sqrt{\lambda}} + x^{-i\sqrt{\lambda}}) = \cos[\sqrt{\lambda}\ln x]$$

and $$\psi_2 = \frac{1}{2i}(x^{i\sqrt{\lambda}} - x^{-i\sqrt{\lambda}}) = \sin[\sqrt{\lambda}\ln x].$$

Then $$\psi = A\cos[\sqrt{\lambda}\ln x] + B\sin[\sqrt{\lambda}\ln x]$$

(1) (b) (continued) We impose the boundary conditions.

$$\psi(1)=0=A, \quad \psi(2)=0=B\sin[\sqrt{\lambda}\ln 2].$$

$B=0$ gives the trivial solution, so we take

$$\sin[\sqrt{\lambda}\ln 2]=0 \Rightarrow \sqrt{\lambda}\ln 2 = n\pi, \ n=1,2,3..$$

and

$$\lambda_n = \frac{n^2\pi^2}{(\ln 2)^2}, \quad \psi_n = \sin\left[n\pi\frac{\ln x}{\ln 2}\right].$$

(c) ~~Eigenfunct~~ Eigenfunctions associated with different eigenvalues should be orthogonal with respect to the weight function $\frac{1}{x}$. We verify this by direct calculation.

$$I=\int_1^2 \psi_n(x)\psi_m(x)\frac{1}{x}\,dx = \int_1^2 \sin\left[n\pi\frac{\ln x}{\ln 2}\right]\sin\left[m\pi\frac{\ln x}{\ln 2}\right]\frac{dx}{x}$$

We make a change of variables in the integral:

$$u=\frac{\ln x}{\ln 2}.$$

Then u runs from 0 to 1, and $\frac{dx}{x}=\ln 2\,du$, so

$$I=\ln 2\int_0^1 \sin(n\pi x)\sin(m\pi x)\,dx = \begin{cases} 0, & m\neq n \\ \frac{1}{2}\ln 2, & m=n \end{cases}.$$

(d) See the Mathematica notebook.

(e)
$$\frac{\ln x}{\ln 2} = \sum_{n=0}^{\infty} C_n \sin\left[n\pi\frac{\ln x}{\ln 2}\right]$$

We use orthogonality to get

$$\int_1^2 \frac{\ln x}{\ln 2}\cdot\frac{1}{x}\cdot\sin\left[n\pi\frac{\ln x}{\ln 2}\right]dx = C_n\int_1^2 \frac{1}{x}\left[\sin\left\{n\pi\frac{\ln x}{\ln 2}\right\}\right]^2 dx$$

(1) (e) (continued) Again we let $U = \ln x/\ln 2$.

$$\ln 2\int_0^1 U\sin(n\pi U)\,dU = C_n\int_0^1 \ln 2\,\sin^2(n\pi U)\,dU$$

$$\text{so}\quad C_n = \frac{\int_0^1 U\sin(n\pi U)\,dU}{\int_0^1 \sin^2(n\pi U)\,dU} = \frac{2(-1)^{n+1}}{n\pi}$$

See Mathematica notebook for the plot.

(2) (a) The problem is inhomogeneous, so we will split it first into steady state (Φ_s) and transient ($\tilde{\Phi}$) parts.

$$\Phi = \Phi_s + \tilde{\Phi}$$

Problem for Φ_s: $\frac{d}{dx}\left(x\frac{d\Phi_s}{dx}\right) - 1 = 0$, $\Phi_s(1) = 1$
$\Phi_s(2) = 1$.

Problem for $\tilde{\Phi}$: $\frac{1}{x}\frac{\partial\tilde{\Phi}}{\partial t} = \frac{\partial}{\partial x}\left(x\frac{\partial\tilde{\Phi}}{\partial x}\right)$, $1<x<2$, $t>0$,

$$\tilde{\Phi}(1,t) = 0,\quad \tilde{\Phi}(2,t) = 0,\quad \tilde{\Phi}(x,0) = x - \Phi_s(x).$$

<u>Solution for Φ_s.</u>

We integrate $\frac{d}{dx}\left(x\frac{d\Phi_s}{dx}\right) = 1$ to get $x\frac{d\Phi_s}{dx} = x + A$, so $\frac{d\Phi_s}{dx} = 1 + \frac{A}{x}$. We integrate again to get $\Phi_s = x + A\ln x + B$.
We impose the BC: $\Phi_s(1) = 1 = 1 + B \Rightarrow B = 0$
$\Phi_s(2) = 1 = 2 + A\ln 2 \quad\Rightarrow A = -\frac{1}{\ln 2}$, so $\Phi_s = x - \frac{\ln x}{\ln 2}$.

(2) (G) (continued).

Solution for $\tilde{\Phi}$

We try $\tilde{\Phi} = F(x)G(t)$. We substitute into the equation to get

$$\frac{1}{G}\frac{dG}{dt} = \frac{x}{F}\frac{d}{dx}\left(x\frac{dF}{dx}\right).$$

The separation has worked. Each side is equal to the same constant which we call $-\lambda$. Then

$$\frac{dG}{dt} = -\lambda G \Rightarrow G(t) = \text{const}\cdot e^{-\lambda t}.$$

$$\frac{d}{dx}\left(x\frac{dF}{dx}\right) = -\frac{\lambda}{x}F, \quad 1<x<2$$

$$F(1)=0, \quad F(2)=0$$

This is problem (1), so $F_n = \psi_n(x) = \sin\left[n\pi\frac{\ln x}{\ln 2}\right]$ and

$$\lambda_n = \frac{n^2\pi^2}{(\ln 2)^2}, \quad n=1,2,3,\ldots$$

Then we use superposition to get

$$\tilde{\Phi}(x,t) = \sum_{n=1}^{\infty} C_n e^{-\lambda_n t}\psi_n(x).$$

We impose the initial condition:

$$\tilde{\Phi}(x,0) = x - \Phi_s(x) = x - \left(x - \frac{\ln x}{\ln 2}\right) = \frac{\ln x}{\ln 2} = \sum_{n=1}^{\infty} C_n \sin\left[n\pi\frac{\ln x}{\ln 2}\right]$$

This is the expansion of problem 1, so $C_n = \frac{2(-1)^{n+1}}{n\pi}$.

Then

$$\Phi(x,t) = x - \frac{\ln x}{\ln 2} + \sum_{n=1}^{\infty}\frac{2(-1)^{n+1}}{n\pi} e^{-\left(\frac{n\pi}{\ln 2}\right)^2 t}\sin\left[n\pi\frac{\ln x}{\ln 2}\right]$$

(2) (b) We have

$$\Phi(x,t) = x - \frac{\ln x}{\ln 2} + \frac{2}{\pi}\left\{ e^{-\frac{\pi^2}{(\ln 2)^2}t} \sin\left(\frac{\pi \ln x}{\ln 2}\right) + e^{-\frac{4\pi^2 t}{(\ln 2)^2}} \sin\left(\frac{2\pi \ln x}{\ln 2}\right) + \cdots \right\}$$

If t is large enough, the second and higher terms in the series for the transient will be negligible compared with the first term. In that case

$$\Phi(x,t) \simeq x - \frac{\ln x}{\ln 2} + \frac{2}{\pi} e^{-\frac{\pi^2 t}{(\ln 2)^2}} \sin\left(\frac{\pi \ln x}{\ln 2}\right)$$

For this to be a good approximation, the ratio of the second term to first term amplitude should be small — call it smaller than or equal to ϵ.

$$\frac{\frac{2}{\pi} e^{-\frac{4\pi^2 t}{(\ln 2)^2}}}{\frac{2}{\pi} e^{-\frac{\pi^2 t}{(\ln 2)^2}}} \le \epsilon \Rightarrow t \ge \frac{(\ln 2)^2}{3\pi^2} \ln\left(\frac{1}{\epsilon}\right)$$

If our criterion of smallness is 1%, then $\epsilon = 10^{-2}$ and

$$t \ge \frac{(\ln 2)^2}{3\pi^2} \ln(10^2) = 0.075$$

For $\epsilon = 10^{-3}$, we get $t \ge 0.112$.

ME 201 / MTH 281
Assignment #6
Problem 1

Part (c)

We will use *Mathematica* to verify the orthogonality. We start by defining the eigenfunctions.

```
ψ[x_, n_] := Sin[(n π Log[x]) / Log[2]]
```

Now we calculate the integral of the product of two eigenfunctions and the weight function.

```
int = Integrate[ψ[x, n] * ψ[x, m] * (1 / x), {x, 1, 2}]
```

$$\frac{\text{Log}[2]\ (n\,\text{Cos}[n\,\pi]\ \text{Sin}[m\,\pi] - m\,\text{Cos}[m\,\pi]\ \text{Sin}[n\,\pi])}{\left(m^2 - n^2\right)\,\pi}$$

```
Simplify[int, {m, n} ∈ Integers]
```

```
0
```

We see that *Mathematica* has been careless. We get 0, which is correct when m and n are different integers, but when m = n, the result is not zero, in spite of what *Mathematica* just said. We set m = n, which should give us the normalization integral.

```
norm = Integrate[ψ[x, n] * ψ[x, n] * (1 / x), {x, 1, 2}]
```

$$-\frac{\text{Log}[2]\ (-2\,n\,\pi + \text{Sin}[2\,n\,\pi])}{4\,n\,\pi}$$

```
norm = Simplify[norm, n ∈ Integers]
```

$$\frac{\text{Log}[2]}{2}$$

This is the same result we found by hand.

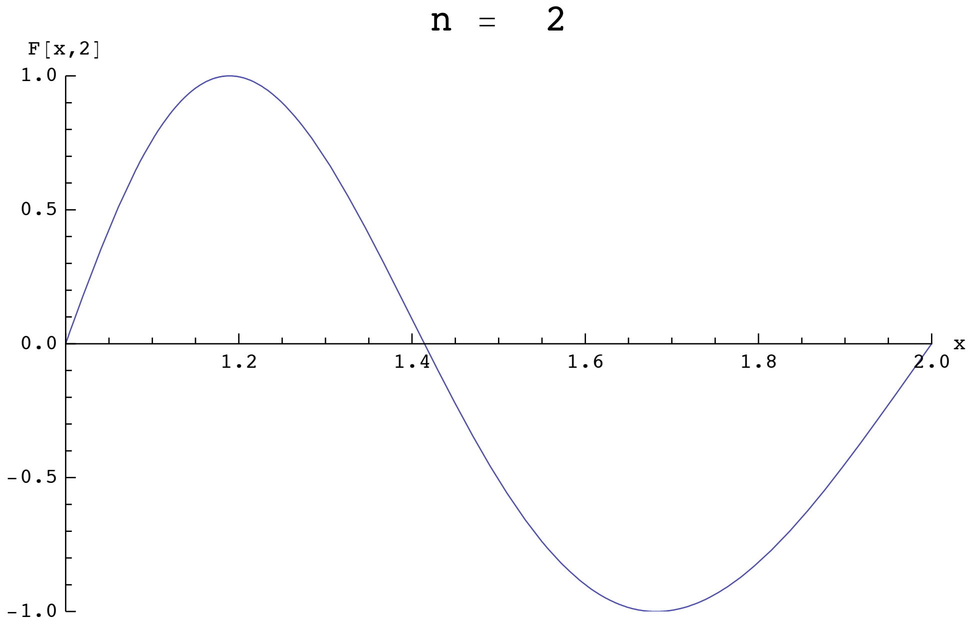
Part (d)

Now we use a Do loop to plot the first five eigenfunctions. We first define a command eigraph[n] which produces a graph of the nth eigenfunction, and then we put that command in the Do loop.

```
eigraph[n_] :=
 Plot[ψ[x, n], {x, 1, 2}, AxesLabel → {"x", Row[{"F[x,", n, "]"}]},
  PlotLabel → Row[{"n =", PaddedForm[n, 2]}],
  PlotRange → {{1, 2}, {-1, 1}}]
```

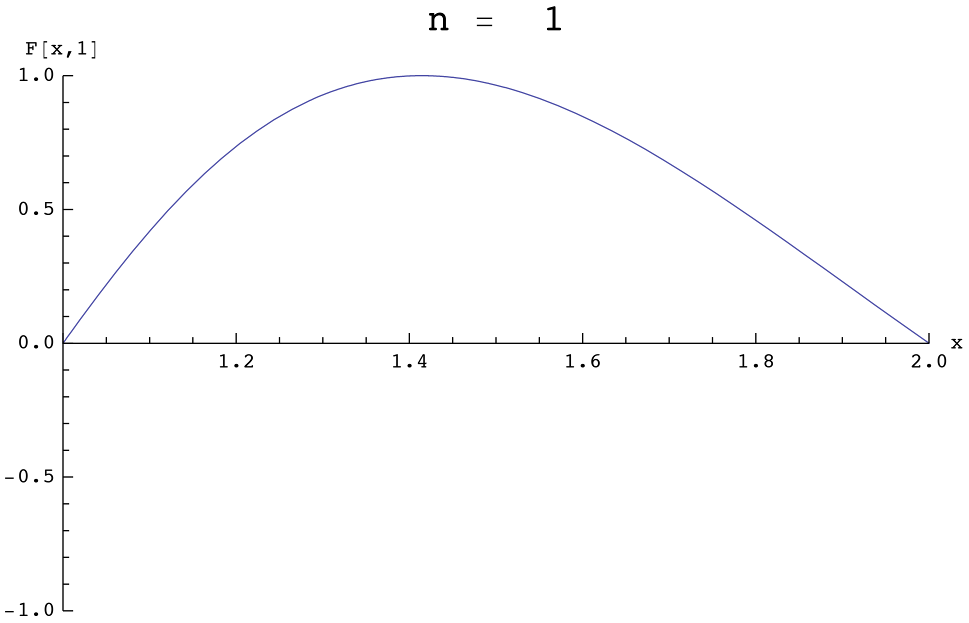
We try this for n = 2.

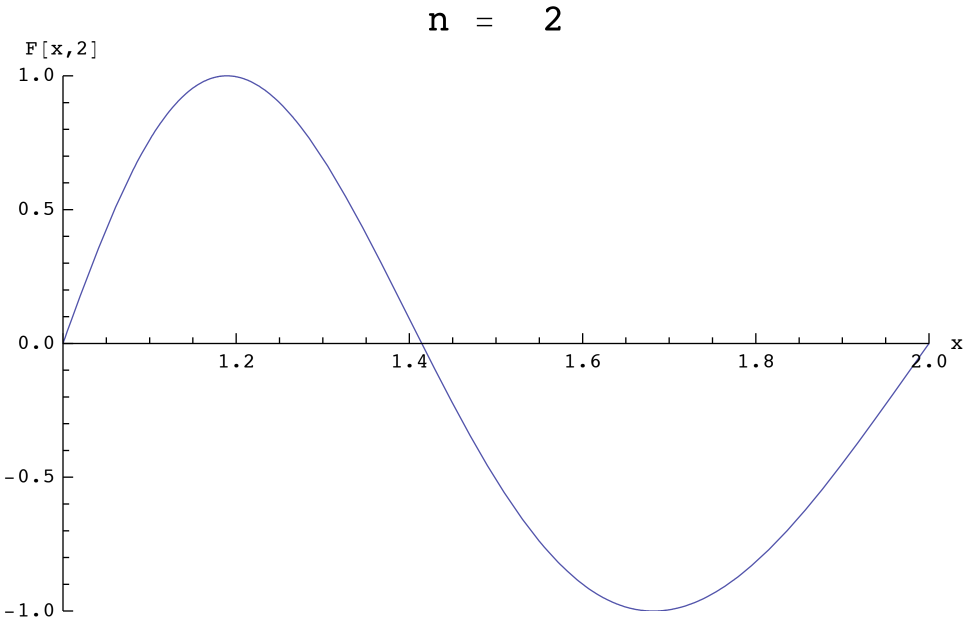
```
eigraph[2]
```

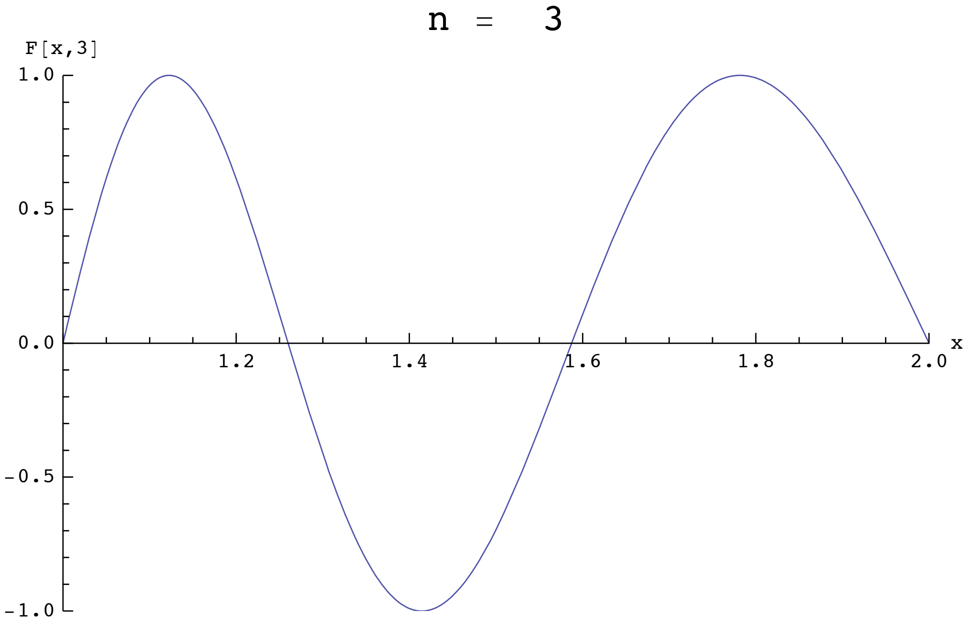


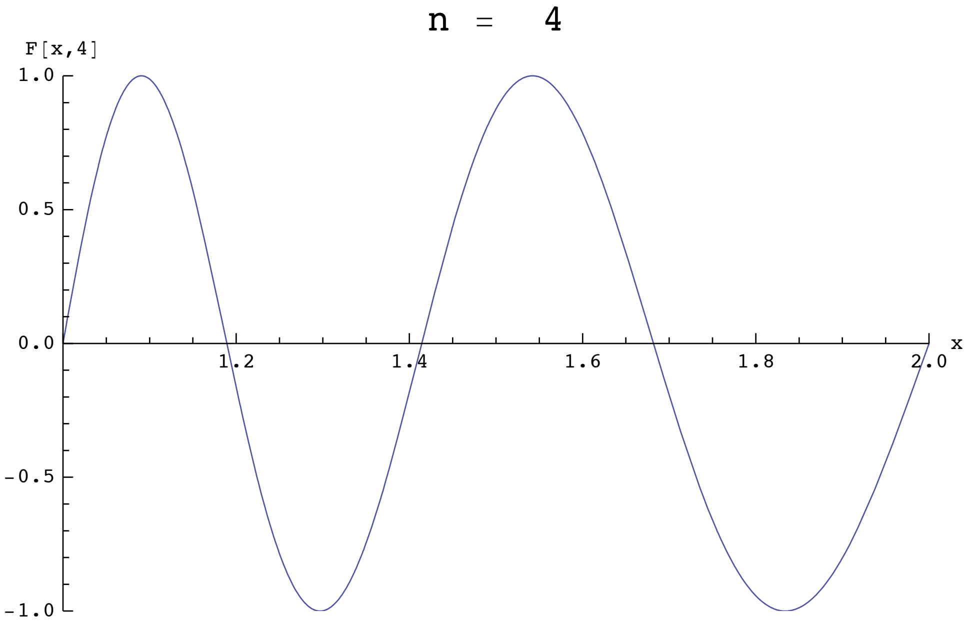
Now the loop.

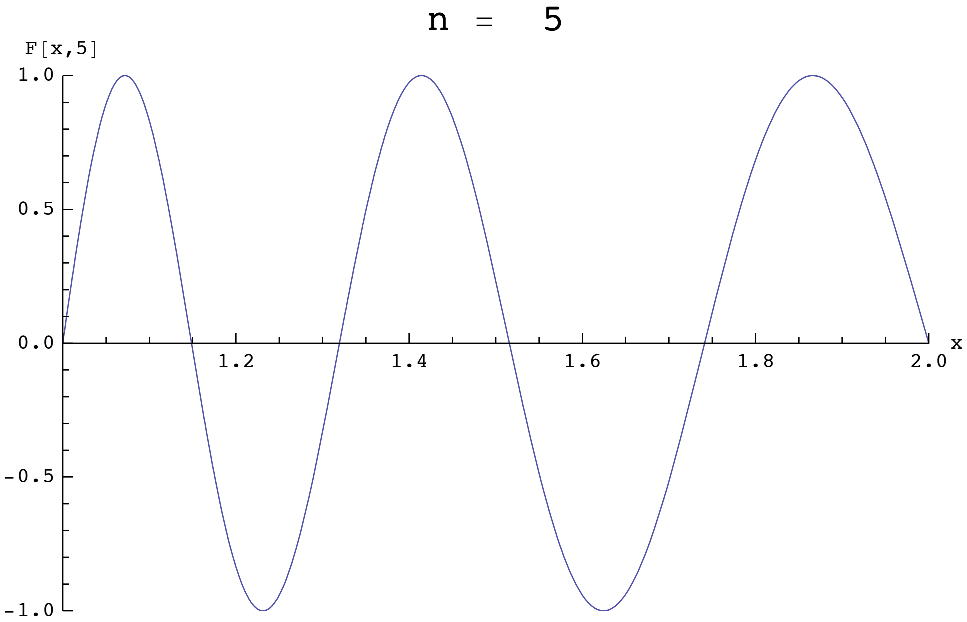
```
Do[Print[eigraph[s]], {s, 1, 5}];
```




n = 2
F[x,2]
1.0
0.5
0.0
-0.5
-1.0
1.2
1.4
1.6
1.8
2.0
x


n = 3
F[x,3]
1.0
0.5
0.0
-0.5
-1.0
1.2
1.4
1.6
1.8
2.0
x





It is easy to see that the nth eigenfunction has exactly n-1 interior zeros.

■ Part (e)

To plot the series, we start by getting the expansion coefficient, which we did earlier by hand.

```
f[x_] := Log[x] / Log[2]
```

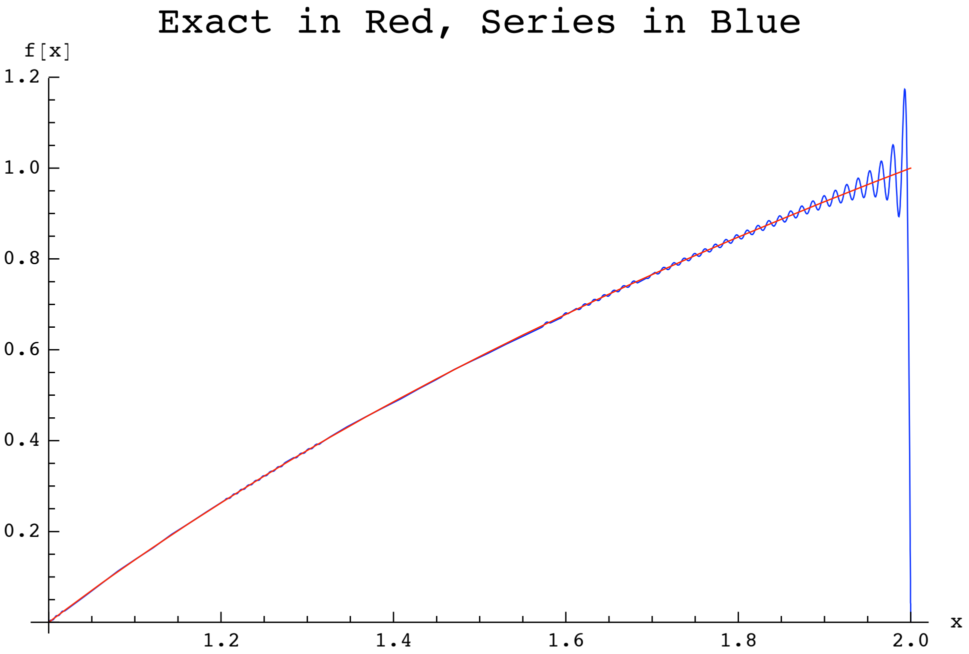
```
c[n_] = Simplify[
  Integrate[ψ[x, n] * f[x] * (1 / x), {x, 1, 2}] / norm, n ∈ Integers]
```

$$-\frac{2\,(-1)^n}{n\,\pi}$$

The series is slowly converging (like 1/n) so we will go up to n = 201.

```
seriesf[x_] := Sum[c[n] * ψ[x, n], {n, 1, 201}]

Plot[{seriesf[x], f[x]}, {x, 1, 2},
 PlotStyle → {RGBColor[0, 0, 1], RGBColor[1, 0, 0]},
 AxesLabel → {"x", "f[x]"},
 PlotLabel -> "Exact in Red, Series in Blue"]
```



We see that the series does reasonably well except for the Gibbs overshoot at x = 2.