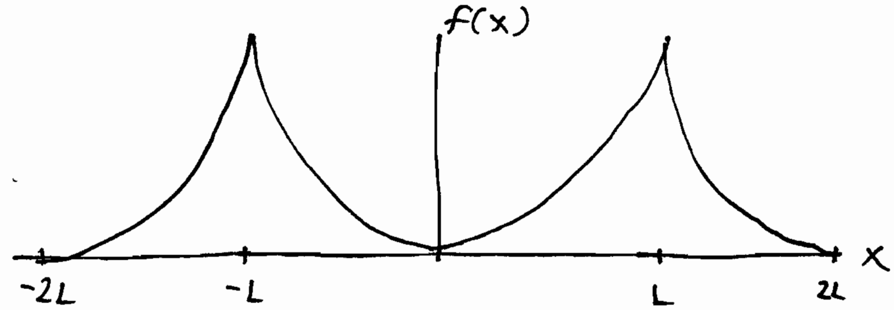
ME 201/MTH 281 ASSIGNMENT # 3 SOLUTIONS SEPT 25, 2008

(1) (a)



(b) The extended function is continuous but its derivative has jumps at $\pm L, \pm 3L, \ldots$. We expect the coefficients to drop off like $1/n^2$ for large n.

(c) The extended function is even, so all of the sine coefficients are zero. For the cosine coefficients we have

$$a_0 = \frac{1}{2L}\int_{-L}^{L} x^4\,dx = \frac{1}{L}\int_0^L x^4\,dx = \frac{L^4}{5}.$$

$$a_n = \frac{1}{L}\int_{-L}^{L} x^4\cos(n\pi x/L)\,dx = \frac{2}{L}\int_0^L x^4\cos(n\pi x/L)\,dx.$$

This last integral by repeated integrations-by-parts, but it is easier to use Mathematica. Either way the result is

$$a_n = \frac{8L^4(-1)^n\left[n^2\pi^2-6\right]}{n^4\pi^4}.$$

(The Mathematica calculations are given in the notebook at the end of these solutions.)

Because extended f is continuous and piecewise smooth, the Fourier series will converge everywhere to extended f. In particular on $-L\le x\le L$,

$$f(x) = x^4 = \frac{L^4}{5} + \frac{8L^4}{\pi^4}\sum_{n=1}^{\infty}\frac{(-1)^n\left[n^2\pi^2-6\right]}{n^4}\cos\left(\frac{n\pi x}{L}\right).$$

(1) (c) (continued) For large n, $n^2\pi^2 \gg 6$, and

$$b_n \simeq \frac{8L^4(-1)^n}{n^2\pi^2},$$

and the coefficients drop off like $\frac{1}{n^2}$ as predicted.

(2) (a) The function is odd so the cosine coefficients are all zero. The sine coefficients are

$$b_n = \frac{1}{L}\int_{-L}^{L} x^3 \sin(n\pi x/L)\,dx = \frac{2}{L}\int_0^L x^3 \sin(n\pi x/L)\,dx.$$

Let $I = \int_0^L x^3 \sin(n\pi x/L)\,dx$. We use integration-by-parts
$u = x^3$, $du = 3x^2 dx$, $dv = \sin(n\pi x/L)dx$, $v = -\frac{L}{n\pi}\cos(n\pi x/L)$.
Then $I = uv - \int v\,du = -\frac{Lx^3}{n\pi}\cos(n\pi x/L)\Big|_0^L + \frac{3L}{n\pi}\int_0^L x^2\cos(n\pi x/L)dx$

or $I = -(-1)^n \frac{L^4}{n\pi} + \frac{3L}{n\pi} J$.

$J = \int_0^L x^2 \cos(n\pi x/L)\,dx$. $u = x^2$, $du = 2x\,dx$, $dv = \cos(n\pi x/L)dx$
$v = \frac{L}{n\pi}\sin(n\pi x/L)$. So $J = \frac{x^2 L}{n\pi}\sin(n\pi x/L) - \frac{2L}{n\pi}\int_0^L x\sin\left(\frac{n\pi x}{L}\right)dx$

or $J = -\frac{2L}{n\pi}K$, where $K = \int_0^L x\sin\left(\frac{n\pi x}{L}\right)dx$.

One last integration by parts. $u = x$, $du = dx$
$dv = \sin\left(\frac{n\pi x}{L}\right)dx$, $v = -\frac{L}{n\pi}\cos\left(\frac{n\pi x}{L}\right)$. Then

$$K = -\frac{Lx}{n\pi}\cos\left(\frac{n\pi x}{L}\right)\Big|_0^L + \frac{L}{n\pi}\int_0^L \cos\left(\frac{n\pi x}{L}\right)dx$$
$$= -\frac{L^2}{n\pi}(-1)^n.$$

Then $b_n = \frac{2}{L}I = \frac{2}{L}\left\{-(-1)^n\frac{L^4}{n\pi} + \frac{3L}{n\pi}J\right\}$

$$= \frac{2}{L}\left\{-(-1)^n\frac{L^4}{n\pi} + \frac{3L}{n\pi}\left(-\frac{2L}{n\pi}K\right)\right\} = \frac{2}{L}\left\{-(-1)^n\frac{L^4}{n\pi} - \frac{6L^2}{n^2\pi^2}\left(-\frac{L^2}{n\pi}(-1)^n\right)\right\}$$

(2) (a) (continued) so

$$b_n = 2L^3(-1)^n\left[\frac{6}{n^3\pi^3} - \frac{1}{n\pi}\right].$$

We get the same result (with much less effort) in the Mathematica notebook.

(b) on $-L\le x\le L$ $\quad x^4 = \frac{L^4}{5} + \sum_{n=1}^{\infty} \frac{8L^4(-1)^n[n^2\pi^2-6]}{n^4\pi^4}\cos\left(\frac{n\pi x}{L}\right).$

We may differentiate this series once. Then

on $-L<x<L$ $\quad 4x^3 = \sum_{n=1}^{\infty} -8L^3(-1)^n\left[\frac{1}{n\pi} - \frac{6}{n^3\pi^3}\right]\sin\left(\frac{n\pi x}{L}\right)$

so $\quad x^3 = \sum_{n=1}^{\infty} 2L^3(-1)^n\left[\frac{6}{n^3\pi^3} - \frac{1}{n\pi}\right]\sin\left(\frac{n\pi x}{L}\right).$

This is the same as in part (a).

(3) We check the continuity of extended f and derivatives.

$f(1) = 0 = f(-1)$ so extended f is continuous

$f'(1) = -2 = f'(-1)$ so extended f' is continuous.

$f''(1) = -6 \neq f''(-1) = 6$, so extended f'' is discontinuous.

By our convergence guideline, the coefficients should drop off like $\frac{1}{n^3}$.
(By direct calculation one can verify this: all a_n's are zero and

$$b_n = \frac{1}{L}\int_{-L}^{L}(x-x^3)\sin(n\pi x/L)\,dx = 2\int_0^1 (x-x^3)\sin(n\pi x)\,dx$$
$$= 6(-1)^{n+1}/(n\pi)^3.\ )$$

(4) We start with $2x = \sum_{n=1}^{\infty} \frac{4(-1)^{n+1}}{n\pi} \sin(n\pi x)$ on $-1<x<1$.

We integrate this from 0 to a generic x, using x' as the variable of integration.

$$\int_0^x 2x'\,dx' = x^2 = \sum_{n=1}^{\infty} \frac{4(-1)^{n+1}}{n\pi} \int_0^x \sin(n\pi x')\,dx'$$

$$\text{so} \quad x^2 = \sum_{n=1}^{\infty} \frac{4(-1)^{n+1}}{n\pi}\left(-\frac{1}{n\pi}\right)\left[\cos(n\pi x) - 1\right]$$

$$= \sum_{n=1}^{\infty} \frac{4(-1)^{n+1}}{n^2\pi^2} + \sum_{n=1}^{\infty} \frac{4(-1)^n}{n^2\pi^2}\cos(n\pi x)$$

By comparing this with the given series for $g(x)=x^2$, we conclude that they are the same if and only if

$$\frac{1}{3} = \sum_{n=1}^{\infty} \frac{4(-1)^{n+1}}{n^2\pi^2}.$$

Thus we conclude that

$$\sum_{n=1}^{\infty} \frac{(-1)^{n+1}}{n^2} = \frac{\pi^2}{12}.$$

See the Mathematica notebook for a Mathematica check of this result.

(5) (a) $\underline{E}_1 \cdot \underline{E}_2 = \underline{i}\cdot\underline{j} + \underline{i}\cdot\underline{k} = 0$, $\underline{E}_1 \cdot \underline{E}_3 = \underline{i}\cdot\underline{j} - \underline{i}\cdot\underline{k} = 0$,
$\underline{E}_2 \cdot \underline{E}_3 = (\underline{j}+\underline{k})\cdot(\underline{j}-\underline{k}) = \underline{j}\cdot\underline{j} - \underline{j}\cdot\underline{k} + \underline{k}\cdot\underline{j} - \underline{k}\cdot\underline{k} = 1-0+0-1=0$

(b) We start with $\underline{A} = C_1\underline{E}_1 + C_2\underline{E}_2 + C_3\underline{E}_3$ and dot it successively with $\underline{E}_1$, $\underline{E}_2$ and $\underline{E}_3$, using orthogonality. Then $\underline{A}\cdot\underline{E}_1 = C_1\underline{E}_1\cdot\underline{E}_1 + C_2\underline{E}_1\cdot\underline{E}_2 + C_3\underline{E}_1\cdot\underline{E}_3 = C_1 E_1^2$, so
$C_1 = \frac{\underline{A}\cdot\underline{E}_1}{E_1^2} = \frac{(\underline{i}+2\underline{j}+3\underline{k})\cdot\underline{i}}{\underline{i}\cdot\underline{i}} = 1$. Similarly $C_2 = \frac{5}{2}$, $C_3 = -\frac{1}{2}$.

(5)(a)(continued) So $\underline{A} = \underline{E}_1 + \frac{5}{2}\underline{E}_2 - \frac{1}{2}\underline{E}_3$.

Check: $\underline{E}_1 + \frac{5}{2}\underline{E}_2 - \frac{1}{2}\underline{E}_3 = \underline{i} + \frac{5}{2}(\underline{j}+\underline{k}) - \frac{1}{2}(\underline{j}-\underline{k})$

$$= \underline{i} + 2\underline{j} + 3\underline{k}.$$

(b) $\underline{A}^2 = 1^2 + 2^2 + 3^2 = 14$, $C_1^2 \underline{E}_1^2 + C_2^2 \underline{E}_2^2 + C_3^2 \underline{E}_3^2$

$$= (1)^2(1)^2 + \left(\frac{5}{2}\right)^2(2) + \left(-\frac{1}{2}\right)^2 2$$
$$= 14$$

(c) $\underline{e}_1 = \dfrac{\underline{E}_1}{|\underline{E}_1|} = \dfrac{\underline{i}}{1} = \underline{i}$

$\underline{e}_2 = \dfrac{\underline{E}_2}{|\underline{E}_2|} = \dfrac{\underline{j}+\underline{k}}{\sqrt{2}}$

$\underline{e}_3 = \dfrac{\underline{E}_3}{|\underline{E}_3|} = \dfrac{\underline{j}-\underline{k}}{\sqrt{2}}.$

Let $\underline{A} = C_1\underline{e}_1 + C_2\underline{e}_2 + C_3\underline{e}_3$. We dot with $\underline{e}$ and use $\underline{e}_1 \cdot \underline{e}_1 = 1$, and orthogonality to get

$$\underline{A} \cdot \underline{e}_1 = C_1, \text{ so } C_1 = \underline{i} \cdot \underline{i} = 1$$

Similarly $C_2 = \underline{A} \cdot \underline{e}_2 = (\underline{i} + 2\underline{j} + 3\underline{k}) \cdot \dfrac{\underline{j}+\underline{k}}{\sqrt{2}} = \dfrac{5}{\sqrt{2}}$,

$C_3 = \underline{A} \cdot \underline{e}_3 = (\underline{i} + 2\underline{j} + 3\underline{k}) \cdot \dfrac{\underline{j}-\underline{k}}{\sqrt{2}} = -\dfrac{1}{\sqrt{2}}.$

Then
$$\underline{A} = \underline{e}_1 + \frac{5}{\sqrt{2}}\underline{e}_2 - \frac{1}{\sqrt{2}}\underline{e}_3$$

$|\underline{A}|^2 = 1^2 + 2^2 + 3^2 = 14$ and $C_1^2 + C_2^2 + C_3^2 = 1 + \frac{25}{2} + \frac{1}{2}$
$= 14.$

(6) (a) Parseval's theorem says that if the Fourier series of $f(x)$ on $-L \le x \le L$ is

$$a_0 + \sum_{n=1}^{\infty} a_n \cos\left(\frac{n\pi x}{L}\right) + b_n \sin\left(\frac{n\pi x}{L}\right),$$

then
$$\int_{-L}^{L} |f(x)|^2 dx = 2L|a_0|^2 + L\sum_{n=1}^{\infty} |a_n|^2 + |b_n|^2.$$

In ~~this~~ this case $L=1$, $a_n = 0$ and $b_n = \frac{12(-1)^{n+1}}{n^3\pi^3}$.

Then
$$\int_{-1}^{1} (x-x^3)^2 dx = \sum_{n=1}^{\infty} \frac{(12)^2}{n^6\pi^6}$$

The integral is

$$\int_{-1}^{1} (x-x^3)^2 dx = 2\int_0^1 (x-x^3)^2 dx = 2\int_0^1 (x^2 - 2x^4 + x^6) dx$$
$$= 2\left[\frac{1}{3} - \frac{2}{5} + \frac{1}{7}\right] = \frac{16}{105}.$$

Then
$$\sum_{n=1}^{\infty} \frac{1}{n^6} = \frac{\pi^6}{144}\,\frac{16}{105} = \frac{\pi^6}{945}.$$

(b) See Mathematica notebook.

(c) See Mathematica notebook.

ME201/MTH 281
Assignment #3 Solutions

Problem 1

We use *Mathematica* to find the coefficients in the Fourier series. We assign the results to the function a[n].

```
a[0] = (1 / (2 L)) Integrate[x^4, {x, -L, L}]
```

$$\frac{L^4}{5}$$

```
int = (1 / ( L)) Integrate[x^4 Cos[(n π x) / L], {x, -L, L}]
```

$$\frac{2 L^4 \left(4 n \pi \left(-6+n^2 \pi^2\right) \text{Cos}[n \pi]+\left(24-12 n^2 \pi^2+n^4 \pi^4\right) \text{Sin}[n \pi]\right)}{n^5 \pi^5}$$

We now simplify this by telling *Mathematica* that *n* is an integer.

```
a[n_] = Simplify[int, n ∈ Integers]
```

$$\frac{8 (-1)^n L^4 \left(-6+n^2 \pi^2\right)}{n^4 \pi^4}$$

Problem 2

We use *Mathematica* to calculate the coefficients that we obtained earlier with integration-by-parts.

```
int = (2 / L) Integrate[x^3 Sin[(n π x) / L], {x, 0, L}]
```

$$-\frac{2 L^3 \left(n \pi \left(-6+n^2 \pi^2\right) \text{Cos}[n \pi]-3 \left(-2+n^2 \pi^2\right) \text{Sin}[n \pi]\right)}{n^4 \pi^4}$$

Now we simplify based on n being an integer.

```
b[n_] = Simplify[int, n ∈ Integers]
```

$$-\frac{2 (-1)^n L^3 \left(-6+n^2 \pi^2\right)}{n^3 \pi^3}$$

This is the same result we obtained earlier.

Problem 3

As a confirmation of our prediction, we use *Mathematica* to calculate the Fourier coefficients. The cosine coefficients are all zero. The sine coefficients are calculated below.

```
Simplify[Integrate[(x - x^3) Sin[n π x], {x, 0, 1}], n ∈ Integers]
```

$$-\frac{6\,(-1)^n}{n^3\,\pi^3}$$

We get the $1/n^3$ drop-off as predicted.

Problem 4

We could explore the question numerically, but of course we can't prove the result that way, although we can obtain a certain engineering certainty about the result. There is also the possibility that *Mathematica* knows the exact result, so we try that first.

```
Sum[(-1)^(n+1)/n^2, {n, 1, ∞}]
```

$$\frac{\pi^2}{12}$$

We got lucky. Evidently *Mathematica* has a built-in table of known sums of certain series.

Exercise: Prove this result by first applying Parseval's Theorem to the Fourier expansion of f(x) = -1 on -1 < x < 0 and f(x) = +1 on 0 < x < 1, to show that the sum over odd n of $1/n^2 = \pi^2/8$, and then using some elementary analysis of the relevant series. (Hint: the sum over even n of $1/n^2$is simply related to the sum over all n.)

Problem 6

(b)

We define a function which takes the number of terms as an argument and returns a 10-digit value of π. Then we can make a few trials to determine the minimum number of terms to give us a result which rounds to 3.14159.

```
piestimate[n_] := N[(945 Sum[1/k^6, {k, 1, n}])^(1/6), 10]
```

We use this to construct a table.

```
Table[{n, piestimate[n]}, {n, 1, 10}]
```

```
{{1, 3.132602581}, {2, 3.140707791},
 {3, 3.141414387}, {4, 3.141540063}, {5, 3.141573003},
 {6, 3.141584035}, {7, 3.141588410}, {8, 3.141590373},
 {9, 3.141591341}, {10, 3.141591856}}
```

For comparison, we look at the 10-digit version of π.

```
N[π, 10]
```

```
3.141592654
```

We see that the minimum number of terms to give us a result which rounds to 3.14159 is 7 -- a small number of terms. Let's see how many terms it takes to match the 10-digit version.

```
Table[{n, piestimate[n]}, {n, 10, 100, 10}]
```

```
{{10, 3.141591856}, {20, 3.141592625},
 {30, 3.141592650}, {40, 3.141592653}, {50, 3.141592653},
 {60, 3.141592653}, {70, 3.141592654}, {80, 3.141592654},
 {90, 3.141592654}, {100, 3.141592654}}
```

```
Table[{n, piestimate[n]}, {n, 60, 70}]
```

```
{{60, 3.141592653}, {61, 3.141592653},
 {62, 3.141592653}, {63, 3.141592653}, {64, 3.141592653},
 {65, 3.141592654}, {66, 3.141592654}, {67, 3.141592654},
 {68, 3.141592654}, {69, 3.141592654}, {70, 3.141592654}}
```

We see that 65 terms is enough to give 10-digit accuracy.

■ (c)

Here we see if *Mathematica* knows the exact result for the sum.

$$\text{Sum}\left[1/n^6, \{n, 1, \infty\}\right]$$

$$\frac{\pi^6}{945}$$

Mathematica does know the result.