

ME 201/MTM 281 ASSIGNMENT # 2 SOLUTIONS SEPT 18, 2008

(1) In class we found the separated solutions which satisfy the equation and the boundary conditions. Then we used superposition to get a solution in the form

$$T(x,t) = \sum_{n=1}^{\infty} C_n e^{-\frac{n^2 \pi^2 \alpha^2 t}{L^2}} \sin\left(\frac{n\pi x}{L}\right).$$

This satisfies the equation and the boundary conditions at  $x=0$  and  $x=L$ . The question is whether we can find values of the coefficients such that the initial condition is satisfied:

$$T(x,0) = \sum_{n=1}^{\infty} C_n \sin\left(\frac{n\pi x}{L}\right) = T_0 \sin\left(\frac{\pi x}{L}\right) + T_1 \sin\left(\frac{\pi x}{L}\right) \cos\left(\frac{2\pi x}{L}\right).$$

If the RHS were a sum of functions  $\sin\left(\frac{n\pi x}{L}\right)$ , we could solve the problem by balancing coefficients. We can use trig identities to put the RHS into that form. We have

$$\sin(\theta_1) \cos(\theta_2) = \frac{1}{2} [\sin(\theta_1 + \theta_2) + \sin(\theta_1 - \theta_2)]$$

$$\text{Then } \sin\left(\frac{\pi x}{L}\right) \cos\left(\frac{2\pi x}{L}\right) = \frac{1}{2} \left[ \sin\left(\frac{3\pi x}{L}\right) + \sin\left(\frac{\pi x}{L}\right) \right]$$

Then to solve the boundary value problem, we

$$\text{need } \sum_{n=1}^{\infty} C_n \sin\left(\frac{n\pi x}{L}\right) = \left(T_0 - \frac{1}{2}T_1\right) \sin\left(\frac{\pi x}{L}\right) + \frac{1}{2}T_1 \sin\left(\frac{3\pi x}{L}\right).$$

We can satisfy this by taking  $C_1 = T_0 - \frac{1}{2}T_1$ ,

$C_3 = \frac{1}{2}T_1$  and all other  $C_n = 0$ . Our solution

(1) (continued) is then

$$T(x,t) = (T_0 - \frac{1}{2}T_1) e^{-\frac{\pi^2 D t}{L^2}} \sin\left(\frac{\pi x}{L}\right) + \frac{1}{2}T_1 e^{-\frac{9\pi^2 D t}{L^2}} \sin\left(\frac{3\pi x}{L}\right).$$

(2) We try  $T = F(x)G(t)$ . We substitute this into the equation:

$$F(x) \frac{dG}{dt} = D \frac{d^2 F}{dx^2} G(t).$$

We divide by  $FGD$ :  $\frac{1}{DG} \frac{dG}{dt} = \frac{1}{F} \frac{d^2 F}{dx^2}.$

The separation has worked, so each side of the equation is equal to the same constant, which we call  $-\lambda$ . Then

$$\frac{1}{DG} \frac{dG}{dt} = -\lambda, \text{ or } \frac{dG}{dt} = -\lambda DG, \text{ the}$$

solution of which is  $G = \text{constant} \times e^{-\lambda Dt}.$

Now the  $x$ -equation.

$$\frac{d^2 F}{dx^2} + \lambda F = 0, \quad 0 < x < L, \quad F(0) = 0, \quad F'(L) = 0$$

The general solution of the equation is

$$F = A \cos \sqrt{\lambda} x + B \sin \sqrt{\lambda} x.$$

We impose the boundary conditions.

$$F(0) = 0 = A$$

$$F'(L) = 0 = B \sqrt{\lambda} \cos \sqrt{\lambda} L.$$

We are temporarily assuming  $\lambda > 0$ , and  $B=0$  gives the trivial solution, so a non-trivial solution requires  $\cos(\sqrt{\lambda} L) = 0$ . The cosine vanishes for argument  $= \frac{\pi}{2}, \frac{3\pi}{2}, \frac{5\pi}{2}, \dots$

hence for  $\sqrt{\lambda} L = (n - \frac{1}{2})\pi, \quad n = 1, 3, 5, \dots$

ME 201/ANTH 201 ASSIGNMENT #2 SOLUTIONS PAGE THREE  
(2) (continued). Then  $\lambda_n = \frac{(\eta - \frac{1}{2})^2 \pi^2}{L^2}$ , and the

Separated solutions are

$$e^{-\frac{(\eta - \frac{1}{2})^2 \pi^2}{L^2} t} \sin\left[(\eta - \frac{1}{2}) \frac{\pi x}{L}\right], \quad \eta = 1, 2, 3, \dots$$

Now we use superposition:

$$T(x, t) = \sum_{\eta=1}^{\infty} C_{\eta} e^{-\frac{(\eta - \frac{1}{2})^2 \pi^2}{L^2} t} \sin\left[(\eta - \frac{1}{2}) \frac{\pi x}{L}\right].$$

The remaining condition in the problem is the initial condition - we want  $T(x, 0) = T_0$ , a constant. We impose this on our solution:

$$T(x, 0) = \sum_{\eta=1}^{\infty} C_{\eta} \sin\left[(\eta - \frac{1}{2}) \frac{\pi x}{L}\right] = T_0.$$

Can we determine the coefficients  $C_{\eta}$  so that this is true? We don't know how to answer that question at this point in our course, so this is as far as we can take the analysis now.

There is one loose end, and that is whether we have found all of the solutions. We assumed  $\lambda > 0$ , and now we need to check the cases  $\lambda = 0$  and  $\lambda < 0$ .

$$\lambda = 0$$

For  $\lambda = 0$ ,  $F'' = 0 \Rightarrow F = Ax + B$ .  $F(0) = 0 = B$   
 $F(L) = 0 = A$ , so we get only the trivial solution.

$$\lambda < 0$$

~~The general solution of the equation is~~  
Let  $\lambda = -\beta$ . Then  $F'' - \beta F = 0$ ,  $\beta > 0$ . The

(2) (continued) general solution is

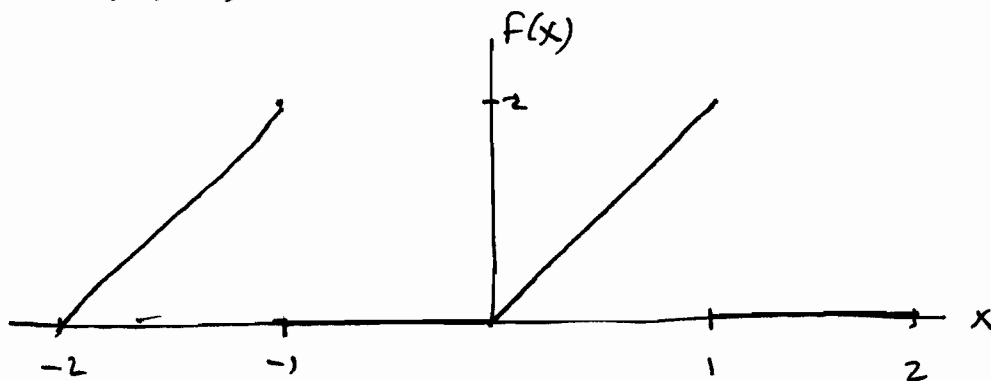
$$F = A \cos \sqrt{\lambda} x + B \sin \sqrt{\lambda} x.$$

We impose the boundary conditions

$$F(0) = A = 0, \quad F'(L) = B \sqrt{\lambda} \cos \sqrt{\lambda} L.$$

The  $\cos$  never vanishes,  $\lambda > 0$  so  $B = 0$  and vice versa we get the trivial solution. We will see later a better way to show that  $\lambda > 0$ .

(3) (a) For  $-1 \leq x \leq 0$ ,  $f(x) = x + (-x) = 0$ , and for  $0 \leq x < 1$ ,  $f(x) = x + x = 2x$ .



(b) On the basic interval  $-1 \leq x \leq 1$ ,  $f$  is continuously differentiable everywhere except at  $x = -1, 0$ , and  $1$ . At all of those points the required limits from the right and left exist:  
 $f(-1-) = 2, f(-1+) = 0, f'(-1-) = 2, f'(-1+) = 0, f(0-) = 0$   
 $f(0+) = 0, f'(0-) = 0, f'(0+) = 2, f(1-) = 2, f(1+) = 0,$   
 $f'(1-) = 2, f'(1+) = 0.$

$$(c) a_0 = \frac{1}{2} \int_{-1}^1 f(x) dx = \frac{1}{2} \int_0^1 2x dx = \frac{1}{2}$$

$$a_n = \int_{-1}^1 f(x) \cos(n\pi x) dx = \int_0^1 2x \cos(n\pi x) dx = \begin{cases} 0, & n \text{ even} \\ -\frac{4}{n^2 \pi^2}, & n \text{ odd} \end{cases}$$

(3) (c) (continued)

$$b_n = \int_{-1}^1 f(x) \sin(n\pi x) dx = \int_0^1 2x \sin(n\pi x) dx$$

$$= \frac{2(-1)^{n+1}}{n\pi}$$

(These integrals can be evaluated by integration-by-parts, or by using Mathematica. We give the Mathematica evaluations in the Mathematica notebook at the end of these solutions.)

The Fourier series is then

$$\frac{1}{2} - \sum_{\substack{n=1 \\ \text{odd}}}^{\infty} \frac{4}{n^2\pi^2} \cos(n\pi x) + \sum_{n=1}^{\infty} \frac{2(-1)^{n+1}}{n\pi} \sin(n\pi x)$$

(d) At  $x = -1$ , the series converges to the mean of the left and right limits, hence to 1. In  $-1 < x < 0$ , the function is continuous so the series converges to the function, namely to 0. At  $x = 0$ , the function is continuous, so the series converges to the function value which is 0. In  $0 < x < 1$ , the function is continuous, so the series converges to the value, which is  $2x$ . At  $x = 1$ , the series converges to the mean of the right and left limits, hence to 1.

(4) See Mathematica notebook

# ME 201/MTH 281

## Assignment #2 Solutions

### Problems 3 and 4

#### ■ Problem 3

We use *Mathematica* to do the integrals defining the Fourier coefficients.

$$a[0] = \frac{1}{2} \int_0^1 2x \, dx$$

$$\frac{1}{2}$$

$$a[n_] = \int_0^1 2x \cos[n\pi x] \, dx$$

$$\frac{2(-1 + \cos[n\pi] + n\pi \sin[n\pi])}{n^2 \pi^2}$$

Now we simplify this by telling *Mathematica* that  $n$  is an integer.

$$a[n_] = \text{Simplify}[a[n], n \in \text{Integers}]$$

$$\frac{2(-1 + (-1)^n)}{n^2 \pi^2}$$

We look at the first two values.

$$a[1]$$

$$-\frac{4}{\pi^2}$$

$$a[2]$$

$$0$$

It is easy to see that all of the coefficients for even  $n$  are zero.

Now we calculate the sine coefficients.

$$b[n_] = \int_0^1 2 x \sin[n \pi x] dx$$

$$\frac{2 (-n \pi \cos[n \pi] + \sin[n \pi])}{n^2 \pi^2}$$

We simplify this using the fact that n is an integer.

`b[n] = Simplify[b[n], n ∈ Integers]`

$$-\frac{2 (-1)^n}{n \pi}$$

---

## ■ Problem 4

### ■ Part a

We begin by defining the base function, which is the expression for the function on the interval [-1,1].

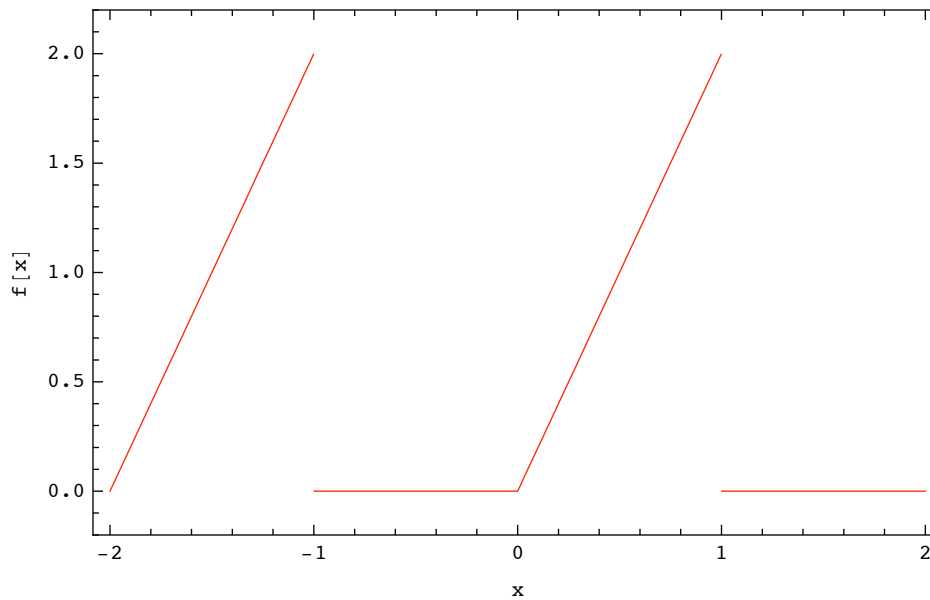
`fbas[x_] := Abs[x] + x`

Now we extend this periodically, using the technique given in the hint.

`f[x_] := fbas[Mod[x, 2, -1]]`

We plot this over the interval [-2,2]. We use red for the color of f[x].

```
graphbase = Plot[f[x], {x, -2, 2}, FrameLabel -> {"x", "f[x]"},
  ImageSize -> 350, PlotStyle -> RGBColor[1, 0, 0],
  Frame -> True, Axes -> False, PlotRange -> {-0.2, 2.2}]
```



### ■ Part b

The Fourier coefficients have been defined above in problem 3.

Now we use the function Sum to define the Mth partial sum of the series, by which we mean the sum of the first M sine and cosine terms. The M=0 partial sum is just the coefficient a[0].

```
partialsum[x_, M_] :=
  a[0] + Sum[(a[n] Cos[n π x] + b[n] Sin[n π x]), {n, 1, M}]
```

We check this by looking at the first two values.

```
partialsum[x, 1]
```

$$\frac{1}{2} - \frac{4 \cos[\pi x]}{\pi^2} + \frac{2 \sin[\pi x]}{\pi}$$

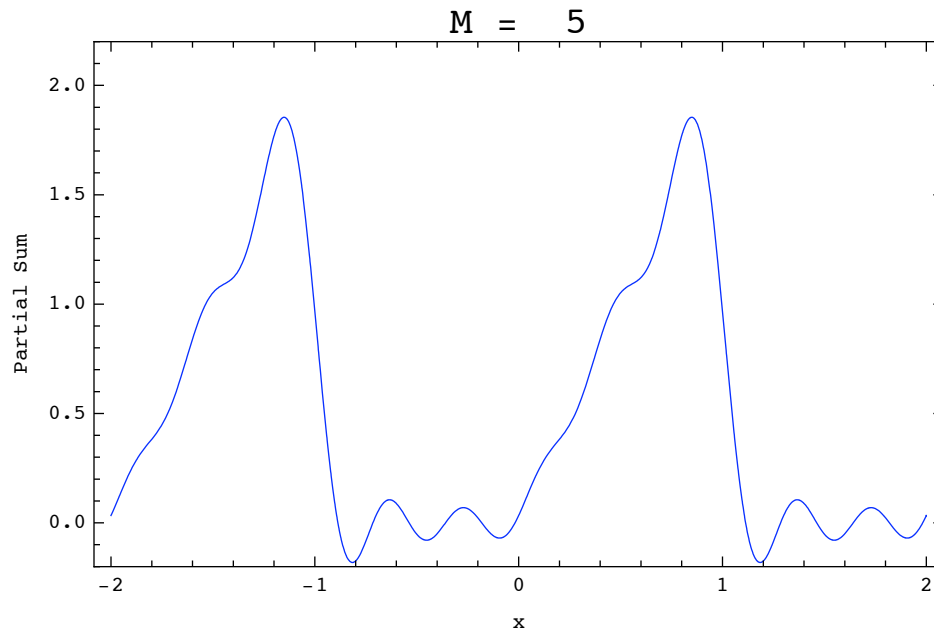
```
partialsum[x, 2]
```

$$\frac{1}{2} - \frac{4 \cos[\pi x]}{\pi^2} + \frac{2 \sin[\pi x]}{\pi} - \frac{\sin[2 \pi x]}{\pi}$$

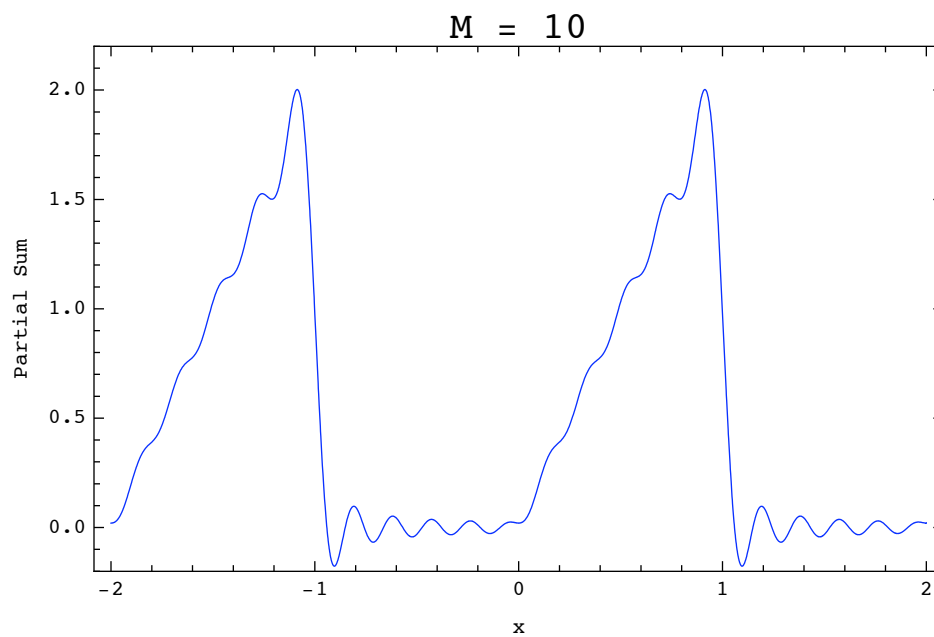
Now we construct plots of the partial sums for M = 5, 10, and 20. After we construct the graphs, we will combine them with the original function to see how well they fit. For this purpose, it is helpful to have a different color, so we plot the partial sums graphs in blue. We define a function which takes M as an argument and produces the graph of the Mth sum. After defining the function, we execute it for M = 5, 10 and 12.

```
graph[M_] :=  
  Plot[partialsum[x, M], {x, -2, 2}, FrameLabel -> {"x", "Partial Sum"},  
    PlotLabel -> Row[{"M =", PaddedForm[M, 2]}], Frame -> True,  
    Axes -> False, PlotStyle -> RGBColor[0, 0, 1],  
    ImageSize -> 350, PlotRange -> {-0.2, 2.2}]
```

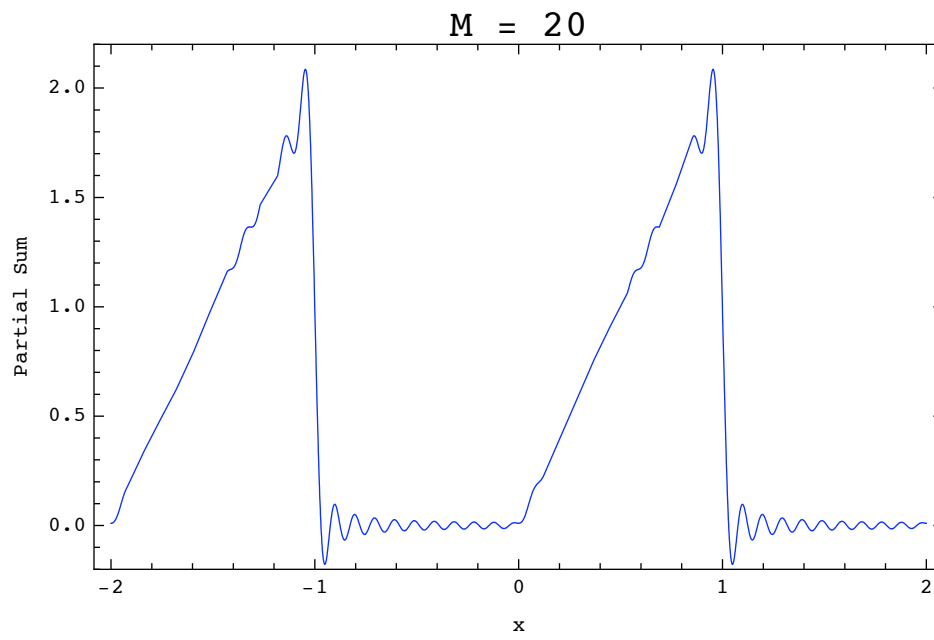
```
sum5 = graph[5]
```



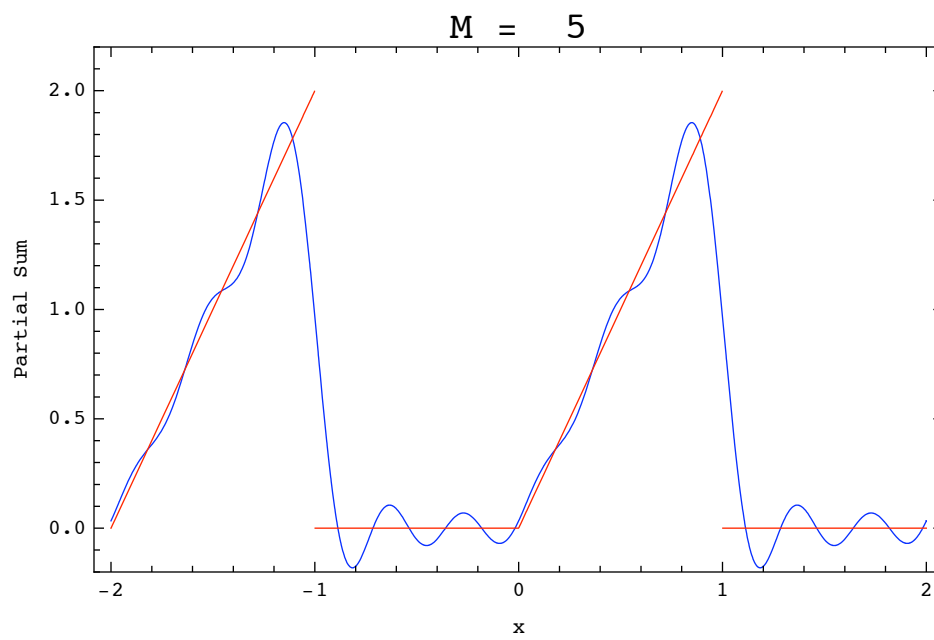
```
sum10 = graph[10]
```



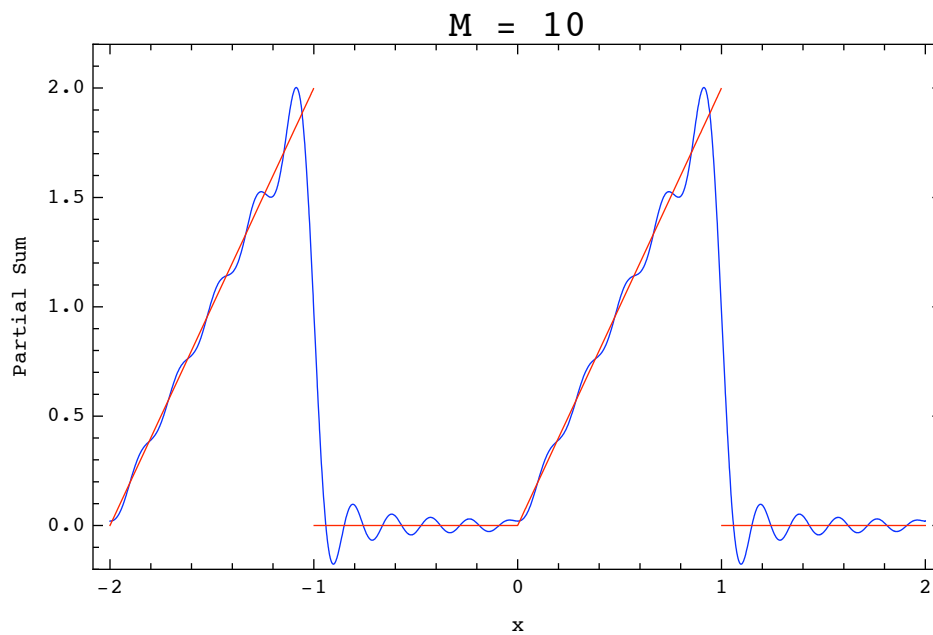
```
sum20 = graph[20]
```



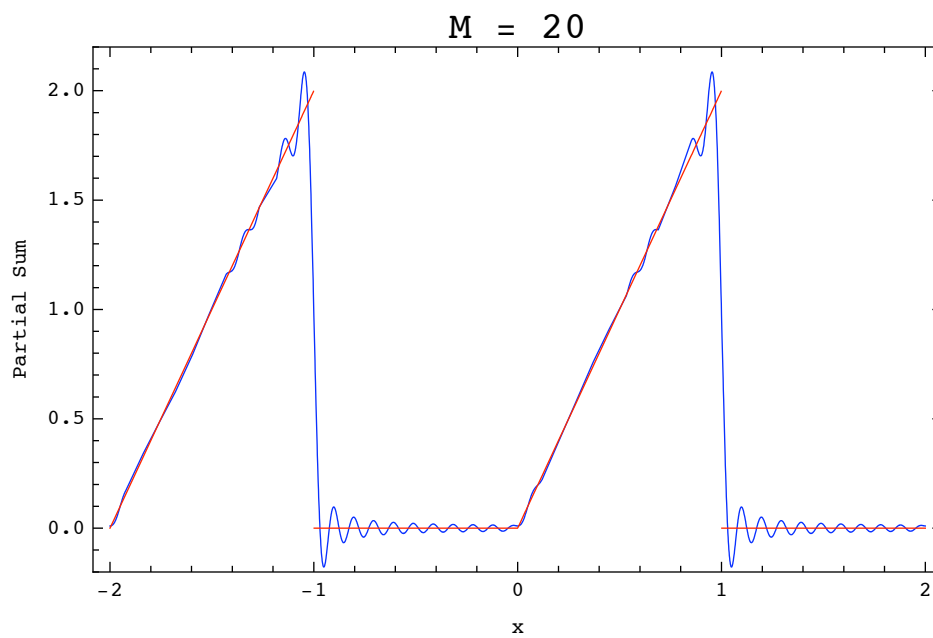
```
Show[sum5, graphbase]
```



Show[sum10, graphbase]



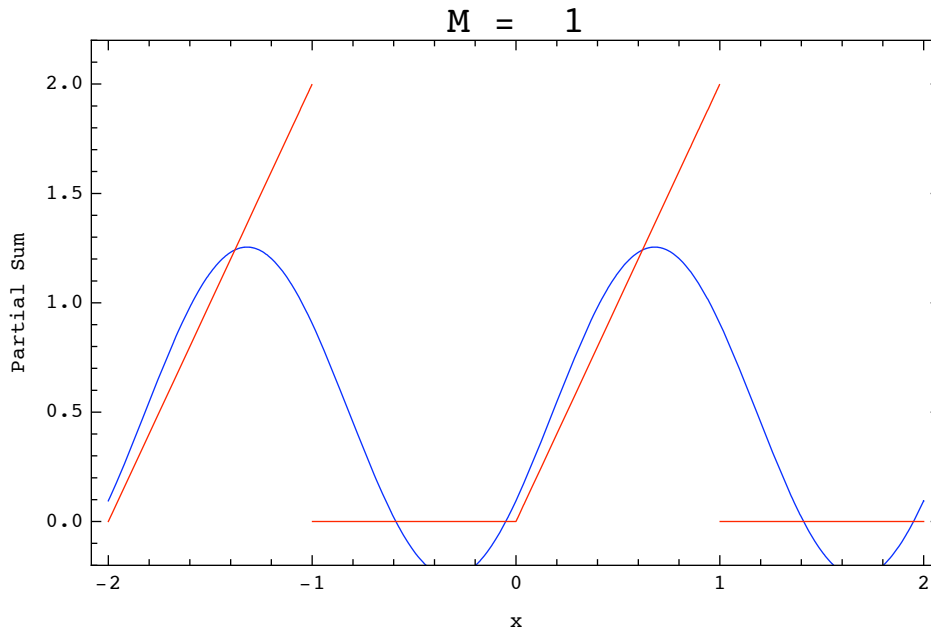
Show[sum20, graphbase]



We see a better representation of the function as we increase the number of terms in the series. Keeping only 5 terms gives a mediocre fit. With 10 terms, the general shape of the function is represented reasonably, and by 20 terms the representation is moderately accurate. As we might have expected, the convergence is worst at the points where the extended  $f$  has discontinuities ( $\pm 1, \pm 3, \dots$ ).

We can make a movie of the convergence by constructing all of the partial sums up to a certain order and then animating the graph sequence. We do this for partial sums up to order 50.

```
Do[Print[Show[graph[i], graphbase]], {i, 1, 50}];
```



The 50 graphs have been collapsed into a single cell so that this notebook won't be so long to print. You can run the movie by selecting the graph sequence, then going to the Menu Graphics-> Rendering-> Animate Selected Graphics.