ME 201/MTH 281 ASSIGNMENT # 11 SOLUTIONS DEC 11, 2008

(1) (a) We try $\Phi = F(r)G(z)$. We substitute into the equation to get

$$\frac{1}{rF}\frac{d}{dr}\left(r\frac{dF}{dr}\right) = -\frac{1}{G}\frac{d^2G}{dz^2} = -\lambda .$$

Then $\frac{d}{dr}\left(r\frac{dF}{dr}\right) + \lambda r F = 0$ and $\frac{d^2G}{dz^2} - \lambda G = 0$.

We expect $\lambda > 0$ so that we get oscillation in r. The solution of the z-equation which satisfies the condition at infinity is $e^{-\sqrt{\lambda} z}$. (The other solution, $e^{\sqrt{\lambda} z}$, blows up as $z \to \infty$.) The r equation is Bessel's equation with a parameter of order 0. The only solution which is well-behaved at $r=0$ is J_0. Then

$$F(r) = J_0(\sqrt{\lambda}\, r).$$

The separated solutions are $e^{-\sqrt{\lambda} z} J_0(\sqrt{\lambda}\, r)$.

(b) The boundary condition is $\Phi|_{r=a} = 0$, so we impose this homogeneous equation on F:

$$F(a) = 0 = J_0(\sqrt{\lambda}\, a).$$

Let $\alpha^{(n)}$ be the nth positive root of J_0. Then

$$\sqrt{\lambda_n}\, a = \alpha^{(n)}$$

$$\text{or} \quad \lambda_n = \frac{{\alpha^{(n)}}^2}{a^2}$$

and $F_n(r) = J_0\left(\alpha^{(n)}\frac{r}{a}\right)$.

(c) The separated solutions $J_0\left(\alpha^{(n)}\frac{r}{a}\right) e^{-\frac{\alpha^{(n)}}{a} z}$ satisfy the equation, the condition as $z \to \infty$, and the boundary condition at $r = a$. The remaining condition is the boundary condition at $z=0$. We superpose the solutions and then impose that condition.

$$\Phi(r,z) = \sum_{n=1}^{\infty} C_n e^{-\frac{\alpha^{(n)}}{a} z} J_0\left(\alpha^{(n)}\frac{r}{a}\right).$$

(1)(c)(continued). We impose the BC at $z=0$:

$$\Phi(r,0)=\Phi_0\left(1-\frac{r^2}{a^2}\right)=\sum_{n=1}^{\infty} C_n J_0\left(\alpha^{(n)}\frac{r}{a}\right).$$

We use orthogonality, remembering that the Bessel functions are orthogonal on $[0,a]$ with respect to the weight function r:

$$C_n=\frac{\Phi_0\int_0^a\left(1-\frac{r^2}{a^2}\right)J_0\left(\alpha^{(n)}\frac{r}{a}\right)r\,dr}{\int_0^a\left[J_0\left(\alpha^{(n)}\frac{r}{a}\right)\right]^2 r\,dr}.$$

We clean up the integrals a bit by making the change of variables $s=r/a$ in both. Then

$$C_n=\Phi_0\frac{\int_0^1(1-s^2)J_0(\alpha^{(n)}s)s\,ds}{\int_0^1[J_0(\alpha^{(n)}s)]^2 s\,ds}$$

These integrals can be done either numerically or analytically. One can show that

$$\int_0^1[J_0(\alpha^{(n)}s)]^2 s\,ds=\frac{1}{2}[J_1(\alpha^{(n)})]^2,$$

and $$\int_0^1(1-s^2)J_0(\alpha^{(n)}s)s\,ds=\frac{4J_1(\alpha^{(n)})}{(\alpha^{(n)})^3}.$$

Then $$C_n=\cancel{\frac{\Phi_0\, 4 J_1(\alpha^{(n)})}{(\alpha^{(n)})^3}}=\frac{8}{J_1(\alpha^{(n)})\,(\alpha^{(n)})^3}.$$

In the Mathematica notebook we will use both numerical integration and the above formula to evaluate C_n.

(d) See Mathematica notebook.

(2) (a) We let $x = \sqrt{\lambda}\, r$. Then $\frac{d}{dr} = \frac{dx}{dr}\frac{d}{dx} = \sqrt{\lambda}\frac{d}{dx}$.

The equation becomes

$$\sqrt{\lambda}\,\frac{d}{dx}\left(\frac{x}{\sqrt{\lambda}}\sqrt{\lambda}\frac{dF}{dx}\right) - \left(\sqrt{\lambda}\,x + \sqrt{\lambda}\frac{\nu^2}{x}\right)F = 0$$

so

$$\frac{d}{dx}\left(x\frac{dF}{dx}\right) - \left(x + \frac{\nu^2}{x}\right)F = 0.$$

In standard form

$$\frac{d^2F}{dx^2} + \frac{1}{x}\frac{dF}{dx} - \left(1 + \frac{\nu^2}{x^2}\right)F = 0.$$

(b) The equation has a regular singular point at $x=0$. We will use the method of Frobenius. We are guaranteed at least one solution of the form

$$F = \sum_{n=0}^{\infty} a_n x^{n+\alpha}.$$

Because $x=0$ is the only singular point in the equation, the series will converge for all x. For the calculation we write the equation in the form

$$x^2F'' + xF' - (x^2+\nu^2)F = 0.$$

We substitute the series for F. The terms are

$$-\nu^2F = -\nu^2\{a_0x^{\alpha} + a_1x^{\alpha+1} \dots\} = \sum_{n=0}^{\infty}(-\nu^2 a_n)x^{\alpha+n}$$

$$-x^2F = -\{a_0x^{\alpha+2} + a_1x^{\alpha+3} \dots\} = \sum_{n=0}^{\infty}(-a_{n-2})x^{\alpha+n}$$

(We are using the convention that $a_n = 0$ for $n<0$.)

$$F' = \alpha a_0x^{\alpha-1} + (\alpha+1)a_1x^{\alpha} + (\alpha+2)a_2x^{\alpha+1} \dots$$

so

$$xF' = \alpha a_0x^{\alpha} + (\alpha+1)a_1x^{\alpha+1} \dots = \sum_{n=0}^{\infty}(\alpha+n)a_nx^{\alpha+n}$$

$$F'' = \alpha(\alpha-1)a_0x^{\alpha-2} + (\alpha+1)(\alpha)a_1x^{\alpha-1} + (\alpha+2)(\alpha+1)a_2x^{\alpha} \dots$$

$$x^2F'' = (\alpha-1)(\alpha)a_0x^{\alpha} + (\alpha)(\alpha+1)a_1x^{\alpha+1} + \dots$$

$$= \sum(\alpha+n-1)(\alpha+n)a_nx^{\alpha+n}$$

(2) (b) (continued). We substitute these into the equation and balance the coefficients of $x^{\alpha+n}$ to get

$$(\alpha+n-1)(\alpha+n)a_n + (\alpha+n)a_n - \nu^2 a_n - a_{n-2} = 0$$

So $$\left[(\alpha+n)^2-\nu^2\right]a_n = a_{n-2}.$$

$n=0$: $$(\alpha^2-\nu^2)a_0 = a_{-2} = 0$$

$\therefore$ $\alpha^2 = \nu^2$, the indicial equation so $\alpha = \pm\nu$

The solution with $\alpha=\nu$ is bounded at $x=0$ (recall that $\nu>0$). If 2ν = integer, there may be only one solution of the Frobenius form and it will be for the larger index, hence $\alpha=\nu$. We set $\alpha=\nu$ and continue with the calculation.

$$\left[(\nu+n)^2-\nu^2\right]a_n = a_{n-2}$$

or

$$n(n+2\nu)a_n = a_{n-2}$$

$n=1$: $(1+2\nu)a_1 = a_{-1} = 0 \Rightarrow a_1 = 0.$

We see from this that all coefficients of odd order will be zero. Continuing we get for $n=2k$

$$2^2 k\left[k+\nu\right]a_{2k} = a_{2k-2}$$

$k=1$: $$2^2(1+\nu)a_2 = a_0$$

$$a_2 = \frac{a_0}{2^2(1+\nu)}$$

(2) (b) (continued)

$$k=2: \quad a_4 = \frac{a_2}{2^2 \cdot 2 \cdot (2+\nu)} = \frac{a_0}{2^4 (2)(1+\nu)(2+\nu)}$$

$$k=3: \quad a_6 = \frac{a_4}{2^2 (3)(3+\nu)} = \frac{a_0}{2^6 (2)(3)(1+\nu)(2+\nu)(3+\nu)}$$

$$k=4: \quad a_8 = \frac{a_6}{2^2 (4)(4+\nu)} = \frac{a_0}{2^8 \cdot 4!\,(1+\nu)(2+\nu)\cdots(4+\nu)}$$

$$\vdots$$

$$a_{2k} = \frac{a_0}{2^{2k}\, k!\,(1+\nu)\cdots(k+\nu)}.$$

Then

$$F(x) = a_0 x^{\nu} \sum_{k=0}^{\infty} \frac{\left(\frac{x}{2}\right)^{2k}}{k!\,(1+\nu)\cdots(k+\nu)}.$$

(c) For $\nu = 1$, the factors in the denominator are $(2)(3)\cdots(k+1) = (k+1)!$ Then for $a_0 = \frac{1}{2}$, we get

$$F(x) = \sum_{k=0}^{\infty} \frac{\left(\frac{x}{2}\right)^{2k+1}}{k!\,(k+1)!}$$

We check the convergence by the ratio test.

$$\left| \frac{\left(\frac{x}{2}\right)^{2k+3}}{(k+1)!\,(k+2)!} \cdot \frac{k!\,(k+1)!}{\left(\frac{x}{2}\right)^{2k+1}} \right| = \frac{\left|\frac{x}{2}\right|^2}{(k+1)(k+2)} \xrightarrow[k\to\infty]{} 0,$$

so the series converges for all x.

(3) Following the hint we try $\Phi(r,\theta,z) = \psi(r,z)\cos\theta$.
We substitute this into

$$\frac{1}{r}\frac{\partial}{\partial r}\left(r\frac{\partial \Phi}{\partial r}\right) + \frac{1}{r^2}\frac{\partial^2 \Phi}{\partial \theta^2} + \frac{\partial^2 \Phi}{\partial z^2} = 0$$

to get $$\frac{1}{r}\frac{\partial}{\partial r}\left(r\frac{\partial \psi}{\partial r}\right) - \frac{\psi}{r^2} + \frac{\partial^2 \psi}{\partial z^2} = 0, \quad \begin{matrix} r<a \\ 0<z<h \end{matrix}$$

The common factor of $\cos\theta$ cancels out.
The boundary conditions on Φ become

$$\psi(r,0)=0, \quad \psi(r,h)=0, \quad \psi(a,z) = \Phi_0.$$

We see from these conditions that the solution should be oscillatory in z. We try

$$\psi = F(r)G(z).$$

We substitute into the equation for ψ to get

$$\frac{1}{F}\left[\frac{1}{r}\frac{d}{dr}\left(r\frac{dF}{dr}\right) - \frac{F}{r^2}\right] = -\frac{1}{G}\frac{d^2G}{dz^2} = \lambda$$

<u>z-equation</u>

$$\frac{d^2G}{dz^2} + \lambda G = 0, \quad 0<z<h, \quad G(0)=0, \quad G(h)=0.$$

$$\therefore \quad \lambda_n = \frac{n^2\pi^2}{h^2}, \quad G_n = \sin\left(\frac{n\pi z}{h}\right).$$

<u>r-equation</u>

$$\frac{d}{dr}\left(r\frac{dF_n}{dr}\right) - \left(\lambda_n r + \frac{1}{r}\right)F_n = 0$$

This is the modified Bessel's equation of order 1 which was solved in problem 2. The solution well-behaved at $r=0$ is

$$F_n(r) = I_1(\sqrt{\lambda_n}\, r) = I_1\left(\frac{n\pi r}{h}\right).$$

(3) (continued) We superpose our solutions.

$$\psi(r,z)=\sum_{n=1}^{\infty} C_n I_1\left(\frac{n\pi r}{h}\right)\sin\left(\frac{n\pi z}{h}\right).$$

We impose the BC at $r=a$:

$$\psi(a,z)=\Phi_0=\sum_{n=1}^{\infty} C_n I_1\left(\frac{n\pi a}{h}\right)\sin\left(\frac{n\pi z}{h}\right)$$

This is a Fourier sine series and we get

$$C_n = \frac{2}{h}\frac{\Phi_0}{I_1\left(\frac{n\pi a}{h}\right)}\int_0^h \sin\left(\frac{n\pi z}{h}\right)dz$$

$$= \frac{4\Phi_0}{n\pi I_1\left(\frac{n\pi a}{h}\right)}, \quad n \text{ odd}$$

$$= 0, \quad n \text{ even}$$

Then

$$\Phi(r,\theta,z) = 4\Phi_0\cos(\theta)\sum_{\substack{n=1\\ n\text{ odd}}}^{\infty}\frac{I_1\left(\frac{n\pi r}{h}\right)}{I_1\left(\frac{n\pi a}{h}\right)}\frac{\sin\left(\frac{n\pi z}{h}\right)}{n\pi}.$$

ME 201/MTH 281
Assignment #11 Solutions

Problem 1 d

We get the first 20 zeros of J0 and put them in the array αzer.

```
αzer = Table[N[BesselJZero[0,i]],{i,1,20}];
```

We define numerically the integrals appearing in the expression for c[[n]]. We start with the numerator.

```
f[r_] := (1 - r^2 / a^2)

num[n_] := Φ0 NIntegrate[f[r] BesselJ[0, αzer[[n]] * r / a] r, {r, 0, a}]
```

Now the denominator.

```
den[n_] := NIntegrate[(BesselJ[0, αzer[[n]] * r / a])^2 r,  {r, 0, a}]

fbc[n_] := num[n] / den[n]
```

We set the parameter values.

```
Φ0 = 100 (** volts **); a = 2 (** m **);
```

Now we calculate the first 20 coefficients and store them in the array coeff.

```
coeff = Module[{ans}, ans = {};
  Do[ans = Append[ans, fbc[n]], {n, 1, 20}]; ans]

{110.802, -13.9778, 4.54765, -2.09909, 1.16362,
 -0.722118, 0.483787, -0.342568, 0.252953, -0.193015,
 0.151221, -0.121077, 0.0987185, -0.0817394, 0.0685835,
 -0.0582113, 0.0499091, -0.0431745, 0.0376465, -0.0330609}
```

As a check on this, we use the analytical formula for the coefficient.

```
fbcanalyt[n_] := Φ0 N[8 / (BesselJ[1, αzer[[n]]] (αzer[[n]])^3)]

coeffanalyt = Module[{ans}, ans = {};
  Do[ans = Append[ans, fbcanalyt[n]], {n, 1, 20}]; ans]

{110.802, -13.9778, 4.54765, -2.09909, 1.16362,
 -0.722118, 0.483787, -0.342568, 0.252953, -0.193015,
 0.151221, -0.121077, 0.0987185, -0.0817394, 0.0685835,
 -0.0582113, 0.0499091, -0.0431745, 0.0376465, -0.0330609}
```

The results are identical to the accuracy shown.

An approximation for large z is obtained by keeping only the first term of the series, and we get

```
coeff[[1]] Exp[-(αzer[[1]] / a) z] BesselJ[0, αzer[[1]] r / a]
```

$110.802\, e^{-1.20241\, z}$ BesselJ[0, 1.20241 r]