ME 201/MTH 281 ASSIGNMENT #10 SOLUTIONS DEC 5, 2008

(1) The separated solutions of $\nabla^2\Phi=0$ which are axisymmetric and valid inside a sphere are

$$r^n P_n(\cos\phi),$$

so we try $\Phi(r,\phi)=\sum C_n r^n P_n(\cos\phi)$. We apply the BC at $r=a$ to determine the C_n's.

$$\Phi(a,\phi)=\sum C_n a^n P_n(\cos\phi) = \Phi_0\cos^3\phi.$$

We could use orthogonality, ~~bu~~ but that is a lot of unnecessary labor. The given function $\cos^3\phi$ is easily expressed in terms of P_n's. We have

$$P_3(\cos\phi)=\frac{1}{2}\left(5\cos^3\phi-3\cos\phi\right),$$

and $P_1(\cos\phi)=\cos\phi$, so

$$\cos^3\phi=\frac{2P_3(\cos\phi)+3P_1(\cos\phi)}{5}.$$

By balancing coefficients in the boundary condition we get

$$C_0=0,\quad C_1 a=\frac{3}{5},\quad C_2=0,\quad C_3a^3=\frac{2}{5}$$ and

all other $C_n=0$. Then

$$\Phi(r,\phi)=\frac{3}{5}\frac{r}{a}P_1(\cos\phi)+\frac{2}{5}\left(\frac{r}{a}\right)^3P_3(\cos\phi)$$

(2) In the region between two concentric spheres, we both the positive and negative powers of r, so the separated solutions are

$$r^n P_n(\cos\phi) \text{ and } r^{-(n+1)}P_n(\cos\phi).$$

(3) (continued) Thus we try

$$\Phi(r,\phi) = \sum_{n=0}^{\infty}\left(a_n r^n + b_n r^{-(n+1)}\right) P_n(\cos\phi)$$

The inner boundary condition is $\Phi(a,\phi) = V = V P_0(\cos\phi)$ and the outer boundary condition is

$$\Phi(b,\phi) = V_1 + V_2\cos^2\phi.$$

We have

$$P_2(\cos\phi) = \frac{1}{2}(3\cos^2\phi - 1)$$

so

$$\cos^2\phi = \frac{1 + 2P_2(\cos\phi)}{3} = \frac{1}{3}P_0 + \frac{2}{3}P_2.$$

Then

$$V_1 + V_2\cos^2\phi = (V_1 + \tfrac{1}{3}V_2)P_0 + \tfrac{2}{3}V_2 P_2.$$

Thus the two boundary conditions only involve P_0 and P_2, so the only terms needed in the series are those for $n=0$ and $n=2$. (If you don't make this observation and you leave the other terms in, the analysis will give you zero coefficients for them.)

So

$$\Phi(r,\phi) = \left(a_0 + \frac{b_0}{r}\right)P_0 + \left(a_2 r^2 + \frac{b_2}{r^3}\right)P_2$$

Now we impose the boundary conditions.

$$r=a: \quad \Phi(a,\phi) = \left(a_0 + \frac{b_0}{a}\right)P_0 + \left(a_2 a^2 + \frac{b_2}{a^3}\right)P_2 = V P_0$$

$$\text{so} \quad a_0 + \frac{b_0}{a} = V, \quad a_2 a^2 + \frac{b_2}{a^3} = 0$$

$$r=b: \quad \Phi(b,\phi) = \left(a_0 + \frac{b_0}{b}\right)P_0 + \left(a_2 b^2 + \frac{b_2}{b^3}\right)P_2 = (V_1 + \tfrac{1}{3}V_2)P_0 + \tfrac{2}{3}V_2 P_2.$$

$$\text{so} \quad a_0 + \frac{b_0}{b} = V_1 + \tfrac{1}{3}V_2, \quad a_2 b^2 + \frac{b_2}{b^3} = \tfrac{2}{3}V_2.$$

(2) (continued) We solve the linear equations for a_0, b_0, a_2, b_2. We get

$$a_0 = \frac{-Va + V_1 b + \frac{1}{3}V_2 b}{b-a}, \quad b_0 = \frac{ab\left(V - V_1 - \frac{1}{3}V_2\right)}{b-a}$$

$$a_2 = \frac{\frac{2}{3}V_2 b^3}{b^5 - a^5}, \quad b_2 = -\frac{\frac{2}{3}V_2 a^5 b^3}{b^5 - a^5}.$$

So

$$\Phi(r,\phi) = \left[\frac{(-Va + V_1 b + \frac{1}{3}V_2 b)}{b-a} + \frac{ab\left(V - V_1 - \frac{1}{3}V_2\right)}{b-a}\frac{1}{r}\right]P_0$$

$$+ \frac{\frac{2}{3}V_2 b^3}{b^5 - a^5}\left[r^2 - \frac{a^5}{r^3}\right]P_2$$

(3) (a) We have $\Phi_0 = V_0 z = V_0 r\cos\phi$, so $\frac{\partial \Phi_0}{\partial r} = V_0\cos\phi$.

The problem for $\tilde{\Phi}$ is then

$$\nabla^2\tilde{\Phi} = 0, \quad r > a, \quad 0 \le \phi \le \pi$$

$$\left.\frac{\partial\tilde{\Phi}}{\partial r}\right|_{r=a} = -V_0\cos\phi = -V_0 P_1(\cos\phi)$$

and $\tilde{\Phi} \to 0$ as $r \to \infty$

The separated solutions of the Laplace equation which go to zero as $r \to \infty$ are

$$\frac{P_n(\cos\phi)}{r^{n+1}}.$$

We try a solution in the form of a superposition of these:

$$\tilde{\Phi}(r,\phi) = \sum_{n=0}^{\infty} \frac{C_n P_n(\cos\phi)}{r^{n+1}}$$

(3)(a) (continued) We impose the BC at $r=a$:

$$\left.\frac{\partial \tilde{\Phi}}{\partial r}\right|_{r=a} = \sum_{n=0}^{\infty} -\frac{(n+1)C_n P_n(\cos\phi)}{a^{n+2}} = -V_0 P_1(\cos\phi)$$

We balance coefficients to get

$$-\frac{(2)C_1}{a^3} = -V_0 \Rightarrow C_1 = \frac{a^3 V_0}{2}$$

all other $C_n = 0$. So

$$\tilde{\Phi}(r,\phi) = \frac{a^3 V_0}{2r^2} P_1(\cos\phi) = \frac{a^3 V_0}{2r^2}\cos\phi$$

and

$$\Phi(r,\phi) = \tilde{\Phi} + \Phi_0 = \frac{a^3 V_0}{2r^2}\cos\phi + V_0 r\cos\phi$$
$$= \left(\frac{a^3}{2r^2} + r\right) V_0 \cos\phi$$

(b) The disturbance velocity vector is $\nabla\tilde{\Phi}$. We have

$$\nabla\tilde{\Phi} = \frac{\partial\tilde{\Phi}}{\partial r}\underline{e_r} + \frac{1}{r}\frac{\partial\tilde{\Phi}}{\partial\phi}\underline{e_\phi}$$
$$= -\frac{a^3 V_0}{r^3}\cos\phi\,\underline{e_r} - \frac{a^3 V_0}{2r^3}\sin\phi\,\underline{e_\phi}$$

Thus the disturbance velocity drops off like $1/r^3$.

(c) To find the speed, we use the total potential

$$\Phi = \left(\frac{a^3}{2r^2} + r\right) V_0 \cos\phi.$$

Then the velocity components are the components of the gradient.

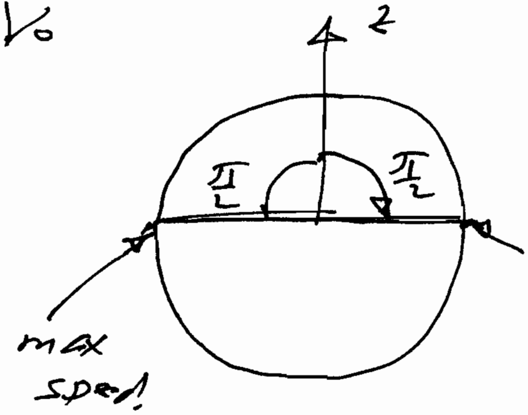
(3)(c) (continued)

$$V_r = \frac{\partial \Phi}{\partial r} = \left(1 - \frac{a^3}{r^3}\right) V_0 \cos\phi \quad = 0 \text{ on } r = a$$

(This is a check that we have correctly satisfied the boundary condition.)

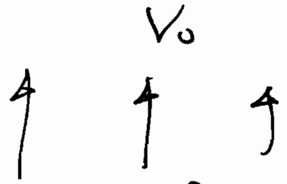
$$V_\phi = \frac{1}{r}\frac{\partial \Phi}{\partial \phi} = -\left(1 + \frac{a^3}{2r^3}\right)\sin\phi \, V_0$$

so $V_\phi \big|_{r=a} = -\frac{3}{2}\sin\phi \, V_0$



The maximum occurs at $\sin\phi = 1$ or $\phi = \pi/2$ which is the equator of the sphere (if we take the poles in the direction of the applied flow)

This speed is 50% greater than the free stream velocity. The physics is basic. The sphere is an obstruction to the flow. The volume rate of flow past the sphere is the same as in unobstructed regions (steady flow, conservation of mass), so the velocity has to be higher near the obstruction.



(d) By the reasoning in part (c), we expect that the fluid has to speed up to get around the obstruction of the car, so we expect to measure a speed greater than 60 mi/hr.