ME 201/MTH 281 ASSIGNMENT #1 SOLUTIONS SEPT 11, 2008

(1) $F = x^2+y^2-2z^2$, $\underline{H} = yz\,\underline{i} - xz\,\underline{j} + z\,\underline{k}$

$$\nabla F = \frac{\partial F}{\partial x}\underline{i} + \frac{\partial F}{\partial y}\underline{j} + \frac{\partial F}{\partial z}\underline{k} = 2x\,\underline{i} + 2y\,\underline{j} - 4z\,\underline{k}$$

$$\nabla \cdot \underline{H} = \frac{\partial H_x}{\partial x} + \frac{\partial H_y}{\partial y} + \frac{\partial H_z}{\partial z} = 0 + 0 + 1 = 1$$

$$\nabla\cdot(F\underline{H}) = \nabla F \cdot \underline{H} + F\,\nabla\cdot\underline{H} = (2x\,\underline{i} + 2y\,\underline{j} - 4z\,\underline{k})\cdot(yz\,\underline{i} - xz\,\underline{j} + z\,\underline{k}) + (x^2+y^2-2z^2)(1)$$

$$= x^2+y^2-6z^2.$$

$$\nabla\times\underline{H} = \left(\frac{\partial H_z}{\partial y} - \frac{\partial H_y}{\partial z}\right)\underline{i} + \left(\frac{\partial H_x}{\partial z} - \frac{\partial H_z}{\partial x}\right)\underline{j} + \left(\frac{\partial H_y}{\partial x} - \frac{\partial H_x}{\partial y}\right)\underline{k} = (0+x)\,\underline{i} + (y-0)\,\underline{j} + (-z-z)\,\underline{k}$$

$$= x\,\underline{i} + y\,\underline{j} - 2z\,\underline{k}$$

(2) The vector ∇T points in the direction of the maximum rate of increase, and $|\nabla T|$ is the maximum rate of increase. A unit vector pointing in the direction of maximum rate of increase is $\nabla T/|\nabla T|$.

$$\nabla T = \nabla F = 2x\,\underline{i} + 2y\,\underline{j} - 4z\,\underline{k}, \text{ so}$$

$$\nabla T\big|_{(1,2,2)} = 2\,\underline{i} + 4\,\underline{j} - 8\,\underline{k}. \text{ Then}$$

$|\nabla T| = \sqrt{4+16+64} = 2\sqrt{21}$, so the maximum rate of increase of temperature is $2\sqrt{21} = 9.17\ {}^\circ C/m$. The direction is

$$\frac{\nabla T}{|\nabla T|} = \frac{1}{\sqrt{21}}(\underline{i} + 2\underline{j} - 4\underline{k}) = 0.218\,\underline{i} + 0.436\,\underline{j} - 0.873\,\underline{k}.$$

(3) We use the divergence theorem, letting V be the volume bounded by S.

$$\oiint_S \underline{H}\cdot\underline{n}\, d\sigma = \iiint_V \nabla\cdot\underline{H}\, d\tau = \iiint_V (1)\, d\tau = \text{volume of ellipsoid}$$

$$= \frac{4}{3}\pi abc.$$

(4) (a) On $z=0$, $\underline{H}=0$, so $\oint_c \underline{H}\cdot d\underline{s}=0$. For the bonus problem we use Stokes' Theorem. Let S be the open surface defined by the portion of $z=1$ for which $x^2+y^2 \le 1$. Then Stokes' Theorem tells us that

$$\oint_D \underline{H}\cdot d\underline{s} = \iint_S \nabla\times\underline{H}\cdot\underline{n}\, d\sigma.$$

On $z=1$, $\nabla\times\underline{H} = x\underline{i}+y\underline{j}-2\underline{k}$. The normal to S is $\underline{n}=\underline{k}$, so $\oint_D \underline{H}\cdot d\underline{s} = \iint_S (-2\underline{k})\cdot\underline{k}\, d\sigma = -2\iint_S d\sigma$.

The area of the circle is π, hence $\oint_D \underline{H}\cdot d\underline{s} = -2\pi$.

(b) $\nabla F\cdot d\underline{s} = dF$, so $\oint_c \nabla F\cdot d\underline{s} = \oint_c dF = 0$ – the change in F around a closed curve is zero.

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(5) We get the general solution by combining the general solution of the homogeneous equation with any particular solution of the inhomogeneous equation.

<u>Homogeneous equation</u>: $\frac{dx}{dt}+2x=0$.

The equation is separable: $\frac{dx}{x} = -2dt$, so

$\ln x = \text{const} - 2t$ hence $x_H = Ce^{-2t}$, where C

(5) (continued) is an arbitrary constant.

Particular Solution

We try $x_p = Ae^t$. We substitute this into the differential equation to get

$$Ae^t + 2Ae^t = e^t \Rightarrow A = \frac{1}{3}.$$

Then the general solution is $x = Ce^{-2t} + \frac{1}{3}e^t$.

We impose the initial condition.

$$x(0) = 0 = C + \frac{1}{3}, \text{ so } C = -\frac{1}{3} \text{ and}$$

$$x(t) = \frac{1}{3}\left(e^t - e^{-2t}\right).$$

(6) To illustrate a different method, we use an integrating factor here. For a linear first order equation in the form

$$\frac{dx}{dt} + a(t)x = b(t),$$

the integrating factor is $\mu(t) = e^{\int a(t)\,dt}$.

In our case

$$\mu(t) = e^{\int 2\,dt} = e^{2t}.$$

We multiply the original equation by e^{2t}.

$$e^{2t}\frac{dx}{dt} + 2e^{2t}x = 1$$

so

$$\frac{d}{dt}\left(xe^{2t}\right) = 1$$

We integrate from $t=0$ to a generic t.

$$x(t)e^{2t} - x(0) = t$$

We ~~are~~ are given $x(0) = 0$, so

$$x(t) = te^{-2t}.$$

(7) The equation is a standard one and has a general solution of the form

$$y(x) = A\cos 3x + B\sin 3x.$$

If you don't recognize the equation, you can always find solutions of such linear constant coefficient equations by assuming a form $y = e^{rx}$, where the constant r is to be determined by substituting the form into the equation:

$$r^2 e^{rx} + 9e^{rx} = 0$$
$$r^2 + 9 = 0$$
$$r^2 = -9$$
$$r = \pm 3i.$$

Thus there are solutions e^{3ix} and e^{-3ix}. For most purposes we prefer real-valued solutions, which we get by linear combinations of these:

$$\tfrac{1}{2}\left(e^{3ix} + e^{-3ix}\right) = \cos 3x$$
$$\frac{1}{2i}\left(e^{3ix} - e^{-3ix}\right) = \sin 3x.$$

We impose the initial conditions on the general solution:

$$\left.\begin{aligned} y(0) &= 1 = A \\ y'(0) &= 3 = 3B \end{aligned}\right\} \Rightarrow A = 1,\ B = 1$$

so

$$y(x) = \cos 3x + \sin 3x.$$

(8) Again the equation is a standard one and has a general solution of the form

$$y(x) = A\cosh 3x + B\sinh 3x.$$

(8) (continued) We derive this. We start by trying an exponential solution $y(x) = e^{rx}$. We substitute this into the equation:

$$r^2 e^{rx} - 9e^{rx} = 0$$
$$r^2 = 9$$
$$r = \pm 3.$$

~~Thus~~ Thus we get solutions e^{3x} and e^{-3x}. For this particular problem, the hyperbolic functions are more convenient, so we use

$$\sinh 3x = \tfrac{1}{2}(e^{3x} - e^{-3x}),$$
$$\cosh 3x = \tfrac{1}{2}(e^{3x} + e^{-3x})$$

Then the general solution is

$$y(x) = A\cosh 3x + B\sinh 3x$$

We impose the initial conditions:

$$y(0) = 0 = A, \quad y'(0) = 3 = 3B$$

So $A=0$, $B=1$ and

$$y(x) = \sinh 3x.$$

(9) (a) We begin with the general solution of the equation: $y(x) = A\cos\sqrt{\lambda}x + B\sin\sqrt{\lambda}x$.

We apply the boundary conditions:

$$y(0) = 1 = A, \quad y(1) = 2 = A\cos\sqrt{\lambda} + B\sin\sqrt{\lambda}$$

So $A=1$, ~~$B = \frac{2 - \cos\sqrt{\lambda}}{\sin\sqrt{\lambda}}$ Thus~~ $B = \dfrac{2-\cos\sqrt{\lambda}}{\sin\sqrt{\lambda}}$

~~$y(x)$~~

$$y(x) = \cos\sqrt{\lambda}x + \frac{2-\cos\sqrt{\lambda}}{\sin\sqrt{\lambda}}\sin\sqrt{\lambda}x.$$

This is a valid solution unless $\sin\sqrt{\lambda} = 0$.

(9) (a) (continued). The zeros of the sine are $\pi, 2\pi, 3\pi, \ldots$ hence $n\pi$ for any positive integer n. Thus the problem has a solution unless $\lambda = n^2\pi^2$ for any positive integer n.

(b) Again we start with $y(x) = A\cos\sqrt{\lambda}x + B\sin\sqrt{\lambda}x$. We impose the boundary conditions.

$$y(0) = 0 = A, \quad y(1) = 0 = B\sin\sqrt{\lambda}.$$

If $B = 0$, we get $y(x) \equiv 0$, the trivial solution. If $B \neq 0$, then $y(1) = 0$ if and only if $\sin\sqrt{\lambda} = 0 \Rightarrow \lambda = n^2\pi^2$ for positive integer n. Thus the only non-trivial solutions occur for $\lambda = n^2\pi^2$.

Exercise: Use your result of part (b) to show that the problem of part (a) has a unique solution when $\lambda \neq n^2\pi^2$.

Later in the course when we study Sturm-Liouville systems, we will develop a general theoretical framework into which the present problem fits.