ME 201/MTH 281 ASSIGNMENT #8 2008

Assignments handed in by 6 PM on Wednesday Nov. 5 will receive a 5 point bonus. Assignments handed in after that but by 6 PM on Thursday Nov. 6 will receive full credit but no bonus. No assignments will be accepted after 6 PM on Nov. 6. This is the last homework assignment before Exam #2 on Nov. 13.

LECTURE SCHEDULE AND READING

Section in Class Notes	Date	Section in Text
V. SEPARATION OF VARIABLES, PART 2		
5.4 Organ Pipe Acoustics	W Oct 29	---
5.5 Standing Acoustic Modes in Three Dimensions	Th Oct 30	7.3,7.4
VI. UNBOUNDED DOMAINS		
6.1 Fourier Integral	F Oct 31	10.1-10.3
6.2 Fourier Transform	M Nov 3	10.3,10.4

PROBLEMS

(1) (30 points) A flexible beam occupies the region $0 \le x \le L$. The ends at $x = 0$ and $x = L$ are simply supported, which means that the ends are fixed in position, but can rotate freely in response to bending moments. If the beam is displaced from equilibrium, it will vibrate, somewhat like a stretched string displaced from equilibrium. The vertical displacement y is a function of both position x, and time t. In solid mechanics (e.g. in ME 226) it is shown that the equation and boundary conditions governing the vibrations are

$$\frac{\partial^2 y}{\partial t^2} = -\sigma \frac{\partial^4 y}{\partial x^4}, 0 < x < L \text{ and } t > 0,$$

$$\text{with } y(0,t) = 0,\ \frac{\partial^2 y}{\partial x^2}(0,t) = 0,\ y(L,t) = 0, \text{ and } \frac{\partial^2 y}{\partial x^2}(L,t) = 0.$$

The parameter σ is a combination of material properties and geometric properties: $\sigma = \dfrac{EI}{\rho A}$.

Here E is the Young's modulus (relating stress and strain), ρ is the material density, A is the cross-sectional area of the beam, and I is a geometric property called the moment of inertia.

(a) (15 points) Find the frequencies of the normal modes of the beam. It is possible to do this by separation of variables, but it is somewhat lengthy. That method will be presented in the solution sheet. Here it is suggested that you try to guess the mode shapes by considering some very familiar functions that are eigenfunctions of $\partial^2/\partial x^2$ and satisfy both boundary conditions at both ends of the beam.

(b) (15 points) Consider a beam with a rectangular cross section with horizontal dimension w = 3 cm and vertical dimension b = 1 mm. For such a section, it can be shown that $I = wb^3/12$. The length of the beam is L = 1 m. The Young's modulus is $E = 2.07 \times 10^{11}$ N/m^2. An observation of the fundamental mode shows that the linear frequency of that mode is 2.33 Hz. What is the density ρ of the material of the beam? What is the linear frequency of the second mode of vibration?

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(2) (30 points)) Consider a vibrating membrane which has a tension T (force per unit length), and a mass per unit area σ. The membrane has a vertical displacement $\Psi(x,y,t)$, where x, y are coordinates in the plane of the membrane and t is time. It can be shown that the equation of motion for small displacements is

$$\frac{\partial^2\Psi}{\partial t^2} = C^2\left(\frac{\partial^2\Psi}{\partial x^2} + \frac{\partial^2\Psi}{\partial y^2}\right), \quad \text{where } C^2 = \frac{T}{\sigma}.$$

The membrane occupies the square region $0 \le x \le a$, $0 \le y \le a$, and the displacement Ψ is zero everywhere along the boundary of this region.

(a) (15 points) If $a = 0.5$ m and $\sigma = 0.1$ kg/m^2, what must the tension T be for the fundamental mode of the membrane to have a frequency one octave below middle C?

(b) (15 points) Use the parameter values of part (a) and the tension you found there to answer the following question. How many modes does the membrane have with frequencies less than or equal to 5000 Hz?

(3) (20 points) Consider the function $f(x) = \begin{Bmatrix} e^{ax} & \text{for } x < 0, \\ xe^{-ax} & \text{for } x \ge 0 \end{Bmatrix}$ where a is a positive constant.

You may use Mathematica to evaluate the integrals in this problem. This will be particularly helpful in part (c).

(a) (7 points) Find the Fourier transform $\tilde{f}(k)$ of $f(x)$ by directly evaluating the integral defining the transform.

(b) (6 points) Verify that $\int_{-\infty}^{\infty} f(x)dx = \tilde{f}(0)$.

(c) (7 points) Verify Parseval's theorem for this transform pair by directly evaluating both of the relevant integrals. Mathematica will be helpful in evaluating the integrals.

(4) (20 points) In this problem, you will apply the Fourier Transform to the solution of a simple ordinary differential equation for a function $y(x)$. The equation is

$$y'' - y = xe^{-|x|}.$$

Here y must satisfy the following additional conditions (these are the conditions that allow the Fourier transform): $y \to 0$ as $x \to \pm\infty$, and y and y' continuous.

(a) (5 points) Take the Fourier transform of the equation and find $\tilde{y}(k)$, the Fourier transform of $y(x)$.

(b) (10 points) Invert the transform to find $y(x)$. Use Mathematica to do the Fourier inversion integral.

(c) (5 points) Plot the solution using Mathematica. Use your graph to estimate the minimum and maximum values of y(x).