ME 201/MTH 281 ASSIGNMENT #5 2008

Assignments handed in by 6 PM on Thursday Oct. 16 will receive a 5-point bonus. Assignments handed in after that but by 4 PM on Friday Oct. 17 will receive full credit but no bonus. No assignments will be accepted after 4 PM on Oct. 17. Note the optional 20-point bonuys problem (5) given on the last page of the assignment.

LECTURE SCHEDULE AND READING

Section in Class Notes	Date	Section in Text
3. Separation of Variables, Part I		
3.2 Laplace Equation	W,Th,F Oct 1,2,3	2.5.1
Fall Break – No Class	M Oct 6	
Exam Review	W Oct 8	---
EXAM #1	Th Oct 9	
3.3 Wave Equation	F Oct 10	4.1-4.4
3.4 Guitar String	M Oct 13	---
3.5 Energy Integrals & Uniqueness	W Oct 15	---

PROBLEMS

3.2 LAPLACE AND LAPLACE-LIKE EQUATIONS

(1) (25 points) Consider the boundary value problem given below for the electrostatic potential in a rectangle. Φ_0 is a constant.

$$\frac{\partial^2\Phi}{\partial x^2}+\frac{\partial^2\Phi}{\partial y^2}=0,\ 0<x<a,\ 0<y<b,$$

$$\text{with } \Phi(x,0)=0, \Phi(x,b)=\Phi_0\frac{x}{a}, \Phi(0,y)=0, \Phi(a,y)=\Phi_0\frac{y}{b},$$

(a) (10 points) Solve this problem by separation of variables and superposition. For (b) and (c), use Mathematica and use the values $a=2$ m, $b=1$ m, and $\Phi_0=20$ volts.
(b) (5 points) Check your calculations by comparing selected numerical values of the boundary potentials $\Phi_0(x/a)$ and $\Phi_0(y/b)$ with values calculated from your series. As a check on your work, show from your solution that the potential at the midpoint of the rectangle is 5.000 volts.
(c) (10 points) Plot on a single contour plot lines of constant voltage for 4, 8, 12, and 16 volts. Use your series to get the values of Φ for the plot.
(d) Verify that the original problem has the closed form solution $\Phi(x,y)=\Phi_0\left(\frac{x}{a}\right)\left(\frac{y}{b}\right)$.

(2) (25 points) Solve the boundary value problem below by separation of variables.

$$4\frac{\partial^2\Phi}{\partial x^2}+\frac{\partial^2\Phi}{\partial y^2}=\alpha\Phi,\quad 0<x<a,\quad 0<y<b,$$

$$\text{with } \Phi(\text{x},0)=0, \Phi(\text{x,b})=0, \Phi(0,\text{y})=0, \text{and } \Phi(a,y)=\Phi_0,$$

where α and Φ_0 are positive constants.

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3.3, 3.4 WAVE EQUATION

(3) (20 points) In class it was shown that the total energy of a stretched string in motion is

$$E[y]=\tfrac{1}{2}\int_0^L\left\{\sigma\left(\frac{\partial y}{\partial t}\right)^2+T\left(\frac{\partial y}{\partial x}\right)^2\right\}dx,$$

where L is the length of the string, T is the tension in the string, and σ is the mass per unit length of the string. We showed that if y is a solution of

$$\frac{\partial^2 y}{\partial t^2}=c^2\frac{\partial^2 y}{\partial x^2}\text{, with } y(0,t)=0 \text{ and } y(L,t)=0,$$

where $c^2=\frac{T}{\sigma}$, then $\frac{dE}{dt}=0$ – that is the energy is conserved. This is consistent with the undamped motion described by the equation, and with the zero-displacement boundary conditions (there are forces on the string at the endpoints, but if there is no motion there, no work is done by the forces). In this problem you will look at the effect of damping on the string energy. We start with the equation including damping. We assume a damping force proportional to the velocity. The vertical force balance then yields the equation

$$\sigma\frac{\partial^2 y}{\partial t^2}=-\varepsilon\frac{\partial y}{\partial t}+T\frac{\partial^2 y}{\partial x^2},$$

where the damping coefficient $\varepsilon>0$ is assumed known. The boundary conditions are

$$y(0,t)=0 \text{ and } y(L,t)=0.$$

Using the expression for the energy, the partial differential equation, and the boundary conditions, show that

$$\frac{dE}{dt}=-\varepsilon\int_0^L\left(\frac{\partial y}{\partial t}\right)^2dx.$$

What can you say in general about the changes of energy with time?

3.5 ENERGY INTEGRALS AND UNIQUENESS

(4) (30 points) Use your result of problem 3 to prove that the solution of the problem given below is unique.

$$\sigma\frac{\partial^2 y}{\partial t^2}=-\varepsilon\frac{\partial y}{\partial t}+T\frac{\partial^2 y}{\partial x^2}+\Gamma(x,t)\text{ , }0<x<L,\ t>0,$$

with $y(0,t)=a(t)$ and $y(L,t)=b(t)$,

and with $y(x,0)=f(x)$ and $\frac{\partial y}{\partial t}(x,0)=g(x)$.

Here a, b, and Γ are given functions. (Hint: Consider the difference of two solutions.)

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BONUS PROBLEM

(5) (20 points) This problem is not required. If you do it, you will get up to 20 extra points. In this problem you will solve the initial value problem for the damped string. The complete problem is given below.

$$\sigma \frac{\partial^2 y}{\partial t^2} = -\varepsilon \frac{\partial y}{\partial t} + T \frac{\partial^2 y}{\partial x^2}, 0 < x < L,\ t > 0,$$

with $y(0,t) = 0$ and $y(L,t) = 0$,

and with $y(x,0) = f(x)$ and $\frac{\partial y}{\partial t}(x,0) = g(x)$.

(a) Although this problem can be solved by separation of variables, you are asked here to use an equivalent but more efficient technique. Based on our previous work with the string problem, we suspect that the functions $\sin[(n\pi x)/L]$ are going to be relevant. Look for a solution in the form

$$y(x,t) = \sum_{n=1}^{\infty} C_n(t) \sin\left(\frac{n\pi x}{L}\right),$$

where the coefficients $C_n(t)$ are to be determined from the equation and the initial conditions. Note that our assumed form already satisfied the boundary conditions. Substitute this form into the partial differential equation, and show that the result is a constant coefficient second order ordinary differential equation for each $C_n(t)$.

(b) In the remainder of the analysis, you may assume that $\varepsilon < \frac{2\pi c\sigma}{L}$, where $c = \sqrt{T/\sigma}$. This inequality guarantees that the oscillatory behavior of each $C_n(t)$ is that of an underdamped oscillator. Find the general solution of the equation for $C_n(t)$, in terms of two arbitrary constants A_n and B_n.

(c) Substitute your result from (b) into the series for y. Now impose the two initial conditions, and use the theory of the sine series to find the coefficients A_n and B_n in terms of the given initial functions *f* and *g*.