

ME 201/MTH 281 ASSIGNMENT #3 2008

Assignments handed in by 6 PM on Wednesday September 24 will receive a 5 point bonus. Assignments handed in after that but by 6 PM on Thursday September 25 will receive full credit but no bonus. No assignments will be accepted after 6 PM on Thursday.

LECTURE SCHEDULE AND READING

<u>Section in Class Notes</u>	<u>Date</u>	<u>Section in Text</u>
2.2 Convergence of Fourier Series	W, Th, F Sept. 17,18, 19	3.2, 3.4, 3.5
2.3 Orthogonality	M Sept. 22	2.3.6

PROBLEMS

2.2 CONVERGENCE OF FOURIER SERIES

(1) (20 points) This problem is concerned with the Fourier series of $f(x) = x^4$ on $-L \leq x \leq L$.

(a) (5 points) Sketch on $-2L \leq x \leq 2L$ the periodically extended function represented by the Fourier series.

(b) (5 points) Based on the smoothness of the periodically extended function, make a prediction about how fast the coefficients drop off with n in the Fourier series for this function.

(c) (10 points) Show that the Fourier series for $f(x) = x^4$ on $-L \leq x \leq L$ is given by

$$f(x) = \frac{L^4}{5} + \sum_{n=1}^{\infty} \frac{8L^4(-1)^n(n^2\pi^2 - 6)}{n^4\pi^4} \cos\left(\frac{n\pi x}{L}\right).$$

You might want to use Mathematica to do the integrals for the Fourier coefficients. Use your answer to check your prediction of part (b).

(2) (15 points)

(a) (10 points) Find the Fourier series for $f(x) = x^3$ on $-L \leq x \leq L$ by calculating the coefficients directly from the defining formulas. (Again you can save some labor by using Mathematica.)

(b) (5 points) Find the Fourier series for $f(x) = x^3$ on $-L \leq x \leq L$ by differentiating termwise the series obtained in problem 1 for $f(x) = x^4$, and show that you get the same result as in part (a).

(3) (15 points) Consider the function $f(x) = x - x^3$ on $-1 \leq x \leq 1$. Without calculating any Fourier coefficients, determine how rapidly the coefficients drop off for the Fourier series of $f(x)$ on $-1 \leq x \leq 1$.

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(4) (15 points) Let $f(x) = 2x$ on $-1 < x \leq 1$, and $g(x) = x^2$ on $-1 < x \leq 1$, with both functions periodically extended with period 2. You may take the two Fourier series below as given.

$$f(x) = \sum_{n=1}^{\infty} \frac{4(-1)^{n+1}}{n\pi} \sin(n\pi x), \text{ and } g(x) = \frac{1}{3} + \sum_{n=1}^{\infty} \frac{4(-1)^n}{n^2\pi^2} \cos(n\pi x) \quad .$$

According to a theorem stated in class on integration of series, we get a valid result when we integrate the series for $f(x)$ termwise. Carry out such an integration for $f(x)$ from $x = 0$ to a

generic x . Because $\int_0^x f(x') dx' = \int_0^x 2x' dx' = x^2$, the result of that integration should be the

Fourier series for $g(x)$. Explore this in detail and show that a byproduct of the calculation is the series given below. Use the Sum function in Mathematica to verify this sum.

$$\sum_{n=1}^{\infty} \frac{(-1)^{n+1}}{n^2} = \frac{\pi^2}{12}.$$

2.3 ORTHOGONALITY

(5) (15 points)

(a) (5 points) Show that the three vectors given below (where \mathbf{i} , \mathbf{j} , and \mathbf{k} are the usual rectangular Cartesian unit vectors) are pairwise orthogonal.

$$\mathbf{E}_1 = \mathbf{i}, \quad \mathbf{E}_2 = \mathbf{j} + \mathbf{k}, \quad \mathbf{E}_3 = \mathbf{j} - \mathbf{k} \quad .$$

(b) (5 points) Let $\mathbf{A} = \mathbf{i} + 2\mathbf{j} + 3\mathbf{k}$. Use orthogonality to find coefficients C_1 , C_2 , and C_3 such that $\mathbf{A} = C_1\mathbf{E}_1 + C_2\mathbf{E}_2 + C_3\mathbf{E}_3$. Verify that $|\mathbf{A}|^2 = C_1^2 |\mathbf{E}_1|^2 + C_2^2 |\mathbf{E}_2|^2 + C_3^2 |\mathbf{E}_3|^2$.

(c) (5 points) By dividing each \mathbf{E} by its length, form a set of three normalized vectors \mathbf{e}_1 , \mathbf{e}_2 , and \mathbf{e}_3 . Use orthogonality to find coefficients c_1 , c_2 , and c_3 such that $\mathbf{A} = c_1\mathbf{e}_1 + c_2\mathbf{e}_2 + c_3\mathbf{e}_3$. Verify that $|\mathbf{A}|^2 = c_1^2 + c_2^2 + c_3^2$.

(6) (20 points) The Fourier series on $-1 \leq x \leq 1$ of $f(x) = x - x^3$ is given by

$$f(x) = \sum_{n=1}^{\infty} \frac{12(-1)^{n+1}}{n^3\pi^3} \sin(n\pi x) \quad .$$

You may take this as given. (You might want to use this to check your answer to problem 3.)

(a) (10 points) Use this series on the interval $[-1, 1]$ and Parseval's theorem to prove that

$$\sum_{n=1}^{\infty} \frac{1}{n^6} = \frac{\pi^6}{945} \quad .$$

(b) (5 points) Use Mathematica and the series of part (a) to calculate a numerical approximation to π . Use enough terms to reproduce the digits 3.14159.

(c) (5 points) Use the Sum function in Mathematica to see if Mathematica knows the exact result for this series.