ME 201/MTH 281 ASSIGNMENT #11 2008 (PRACTICE)

This assignment is a practice assignment for the material on cylindrical coordinates. Assignment solutions will be posted on the web on Thursday Dec. 11.

LECTURE SCHEDULE AND READING

Section in Class Notes	Date	Section in Text
VIII. PROBLEMS IN CYLINDRICAL COORDINATES		
8.1 Singular Points and the Method of Frobenius	M Dec 1	
8.2 Bessel's Equation	W, Th Dec 3, 4	7.7
8.3 Fourier-Bessel Expansions	F, M Dec 5, 8	7.8, 7.9
8.4 Laplace Equation in a Circular Cylinder	W Dec 10	7.9
8.5 Waves in a Coffee Cup	Th Dec 11	

PROBLEMS

(1) (35 points) In this problem, you are to solve the Laplace equation for the electrostatic potential in a circular cylinder of radius a and of semi-infinite height. The potential is given to be zero on the sides of the cylinder, and asymptotically zero as z goes to infinity. The potential on the bottom of the cylinder ($z = 0$) is a specified function of r. Then by symmetry there is no θ dependence in the problem. Here is the complete mathematical formulation:

$$\frac{1}{r}\frac{\partial}{\partial r}\left(r\frac{\partial\Phi}{\partial r}\right)+\frac{\partial^2\Phi}{\partial z^2}=0\ ,0<r<a\ ,0<z<\infty\ ,$$

$$\text{with } \Phi(r,0)=\Phi_0(1-r^2/a^2)\ ,\Phi(a,z)=0\ ,\text{and } \Phi\to 0 \text{ as } z\to\infty\ .$$

Here Φ_0 is a constant.

(a) Look for separated solutions and show that they have the form $e^{-\sqrt{\lambda}z}J_0(\sqrt{\lambda}r)$, where λ is the separation constant.

(b) Show that the eigenvalues are given by $\lambda_n=\left(\alpha^{(n)}/a\right)^2$, where $\alpha^{(n)}$ is the nth positive root of the Bessel function J_0.

(c) Show that the solution for Φ is given by

$$\Phi(r,z)=\sum_{n=1}^{\infty}C_n e^{-\alpha^{(n)}z/a}J_0(\alpha^{(n)}r/a)\ ,\ \text{where } C_n=\Phi_0\frac{\int_0^a(1-r^2/a^2)J_0(\alpha^{(n)}r/a)rdr}{\int_0^a[J_0(\alpha^{(n)}r/a)]^2rdr}\ .$$

(d) For $a = 2$ m and $\Phi_0 = 100$ volts, find the first 20 C_n's. The handout on Bessel functions has some helpful code for this purpose. Show that an approximate solution for large z is given by $\Phi(r,z)\approx 110.802\,e^{-1.20242\,z}J_0(1.20242\,r)$.

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(2) (30 points). The equation given below is called the modified Bessel's equation. You may assume that the parameter λ and the order ν are positive. The function you construct here will be used in problem 3.

$$\frac{d}{dr}\left(r\frac{dF}{dr}\right)-(\lambda r+\frac{\nu^2}{r})F=0.$$

(a) Scale out the parameter by introducing $x=\sqrt{\lambda}r$.

(b) Use the method of Frobenius to find a solution which is well-behaved at $x = 0$.

(c) Show that for the special case ν = 1, your solution may be put into the form given below, which is the standard definition of the modified Bessel function of the first kind of order one.

$$I_1(x)=\sum_{k=0}^{\infty}\frac{\left(\frac{x}{2}\right)^{2k+1}}{k!(k+1)!} \quad .$$

(3) (35 points) Solve the boundary value problem given below for the potential in a circular cylinder of radius *a* and height *h*. Here *r*, θ and *z* are the usual cylindrical coordinates.

$$\nabla^2\Phi=0\ , r<a\ , 0\le\theta\le 2\pi\ , 0<z<h\ ,$$
$$\text{with } \Phi(r,\theta,0)=0\ , \Phi(r,\theta,h)=0\ , \text{and } \Phi(a,\theta,z)=\Phi_0\cos(\theta)\ ,$$

where Φ_0 is a constant. Hint: Separation of variables will work here but it is a long process. If we think of the angular dependence being expressed in terms of a Fourier series, the relevant functions are {1, $\cos(n\theta)$, $\sin(n\theta)$}. The boundary condition suggests strongly that in this particular case only the function $\cos(\theta)$ is needed. On this basis, try a solution in the form

$$\Phi(r,\theta,z)=\Psi(r,z)\cos(\theta)\ .$$

Reformulate the problem in terms of $\Psi(r,z)$. Solve the reformulated problem by separation of variables. Your solution will involve a Fourier sine series in *z*, and the Bessel function I_1 of problem 2.