3/1 INTRODUCTION

The subject of statics deals primarily with the description of the conditions of force which are both necessary and sufficient to maintain the state of equilibrium of engineering structures. This chapter on equilibrium, therefore, constitutes the most central part of statics and should be mastered thoroughly. We shall make continuous use of the concepts developed in Chapter 2 involving forces, moments, couples, and resultants as we apply the principles of equilibrium. The procedures which we shall develop in Chapter 3 constitute a comprehensive introduction to the approach used in the solution of countless problems in mechanics and in other engineering areas as well. This approach is basic to the successful mastery of statics, and the student is urged to read and study the following articles with special effort and attention to detail.

When a body is in equilibrium, the resultant of all forces acting on it is zero. Thus, the resultant force $\mathbf{R}$ and the resultant couple $\mathbf{M}$ are both zero, and we have the equilibrium equations

$$
\mathbf{R} = \Sigma \mathbf{F} = 0 \quad \mathbf{M} = \Sigma \mathbf{M} = 0
$$

(3/1)

These requirements are both necessary and sufficient conditions for equilibrium.

Whereas all physical bodies are inherently three-dimensional, many of them may be treated as two-dimensional when the forces to which they are subjected act in a single plane or may be projected onto a single plane. When this simplification is not possible, the problem must be treated as three-dimensional. We shall follow the arrangement used in Chapter 2 and discuss in Section A the equilibrium of bodies subjected to two-dimensional force systems and in Section B the equilibrium of bodies subjected to three-dimensional force systems.
SECTION A. EQUILIBRIUM IN TWO DIMENSIONS

3/2 MECHANICAL SYSTEM ISOLATION

Before we apply Eqs. 3/1, it is essential that we define unambiguously the particular body or mechanical system to be analyzed and represent clearly and completely all forces which act on the body. The omission of a force or the inclusion of a force which does not act on the body in question will give erroneous results.

A mechanical system is defined as a body or group of bodies which can be isolated from all other bodies. Such a system may be a single body or a combination of connected bodies. The bodies may be rigid or nonrigid. The system may also be a defined fluid mass, liquid or gas, or the system may be a combination of fluids and solids. In statics we direct our attention primarily to a description of the forces which act on rigid bodies at rest, although consideration is also given to the statics of fluids. Once we reach a decision about which body or combination of bodies is to be analyzed, then this body or combination treated as a single body is isolated from all surrounding bodies. This isolation is accomplished by means of the free-body diagram, which is a diagrammatic representation of the isolated body or combination of bodies treated as a single body, showing all forces applied to it by mechanical contact with other bodies which are imagined to be removed. If appreciable body forces are present, such as gravitational or magnetic attraction, then these forces must also be shown on the diagram of the isolated body. Only after such a diagram has been carefully drawn should the equilibrium equations be written. Because of its critical importance we emphasize here that

the free-body diagram is the most important single step in the solution of problems in mechanics.

Before we attempt to draw free-body diagrams, the mechanical characteristics of force application must be recognized. In Art. 2/2 the basic characteristics of force were described, with primary attention focused on the vector properties of force. We noted that forces are applied both by direct physical contact and by remote action and that forces may be either internal or external to the body under consideration. We further observed that the application of external forces is accompanied by reactive forces and that both applied and reactive forces may be either concentrated or distributed. Additionally the principle of transmissibility was introduced, which permits the treatment of force as a sliding vector as far as its external effects on a rigid body are concerned. We will now use these characteristics of force in developing the analytical model of an iso-
MENSIONS

We define a system to be analyzed which act on the force which does results.

A group of bodies system may be. The bodies may tend fluid mass, or fluids and to a description a consideration each a decision analyzed, then is isolated from a means of presentation as a single body, tact with other able body force, then these actuated body. Only the equilibrium we

nt single step

the mechanical zed. In Art. 2/2 with primary at-

We noted that and by remote normal to the body e application of nd that both ap-
d or distributed. produced, which as far as its ex-
ll now use these model of an iso-

lated mechanical system to which the equations of equilibrium will then be applied.

Figure 3/1 shows the common types of force application on mechanical systems for analysis in two dimensions. In each example the force exerted on the body to be isolated by the body to be removed is indicated. Newton’s third law, which notes the existence of an equal and opposite reaction to every action, must be carefully observed. The force exerted on the body in question by a contacting or supporting member is always in the sense to oppose the movement of the body which would occur if the contacting or supporting member were removed.

In example 1 the action of a flexible cable, belt, rope, or chain on the body to which it is attached is depicted. Because of its flexibility, a rope or cable is unable to offer any resistance to bending, shear, or compression and therefore exerts a tension force in a direction tangent to the cable at its point of attachment. The force exerted by the cable on the body to which it is attached is always away from the body. When the tension $T$ is large compared with the weight of the cable, we may assume that the cable forms a straight line. When the cable weight is not negligible compared with its tension, the sag of the cable becomes important, and the tension in the cable changes direction and magnitude along its length. At its attachment the cable exerts a force tangent to itself.

When the smooth surfaces of two bodies are in contact, as in example 2, the force exerted by one on the other is normal to the tangency of the surfaces and is compressive. Although no actual surfaces are perfectly smooth, we are justified in making this assumption for practical purposes in many instances.

When mating surfaces of contacting bodies are rough, example 3, the force of contact may not necessarily be normal to the tangent to the surfaces but may be resolved into a tangential or frictional component $F$ and a normal component $N$.

Example 4 illustrates a number of forms of mechanical support which effectively eliminate tangential friction forces, and here the net reaction is normal to the supporting surface.

Example 5 shows the action of a smooth guide on the body it supports. Resistance parallel to the guide is absent.

Example 6 illustrates the action of a pin connection. Such a connection is able to support force in any direction normal to the axis of the pin. We usually represent this action in terms of two rectangular components. The correct sense of these components in an actual problem will depend on how the member is loaded. When not otherwise initially known, the sense is arbitrarily assigned. Upon computation a positive algebraic sign for the component indicates that the assigned sense is correct. A negative sign indicates the sense is opposite to that assigned.

If the joint is free to turn about the pin, only the force $R$ can
### Modeling the Action of Forces in Two-Dimensional Analysis

<table>
<thead>
<tr>
<th>Type of Contact and Force Origin</th>
<th>Action on Body to be Isolated</th>
</tr>
</thead>
<tbody>
<tr>
<td><strong>1. Flexible cable, belt, chain, or rope</strong></td>
<td></td>
</tr>
<tr>
<td>Weight of cable negligible</td>
<td>Force exerted by a flexible cable is always a tension away from the body in the direction of the cable.</td>
</tr>
<tr>
<td>Weight of cable not negligible</td>
<td></td>
</tr>
<tr>
<td><strong>2. Smooth surfaces</strong></td>
<td>Contact force is compressive and is normal to the surface.</td>
</tr>
<tr>
<td><strong>3. Rough surfaces</strong></td>
<td>Rough surfaces are capable of supporting a tangential component $F$ (frictional force) as well as a normal component $N$ of the resultant contact force $R$.</td>
</tr>
<tr>
<td><strong>4. Roller support</strong></td>
<td>Roller, rock, or ball support transmits a compressive force normal to the supporting surface.</td>
</tr>
<tr>
<td><strong>5. Freely sliding guide</strong></td>
<td>Collar or slider free to move along smooth guides; can support force normal to guide only.</td>
</tr>
</tbody>
</table>

Figure 3/1

be supported. If the joint is not free to turn, a resisting couple $M$ may also be supported. Again, the sense of $M$ is arbitrarily shown here, and in an actual problem will depend on how the member is loaded.

Example 7 shows the resultants of the rather complex distribution of force over the cross section of a slender bar or beam at a built-in or fixed support. The sense of the reactions $F$ and $V$ and the bending couple $M$ will, of course, depend in a given problem on how the member is loaded.

One of the most common forces is that due to gravitational attraction, example 8. This force affects all elements of mass of a
### Modeling the Action of Forces in Two-Dimensional Analysis (cont.)

<table>
<thead>
<tr>
<th>Type of Contact and Force Origin</th>
<th>Action on Body to be Isolated</th>
</tr>
</thead>
<tbody>
<tr>
<td>6. Pin connection</td>
<td>A freely hinged pin connection is capable of supporting a force in any direction in the plane normal to the axis; usually shown as two components $R_x$ and $R_y$. A pin not free to turn may also support a couple $M$.</td>
</tr>
<tr>
<td>![Pin connection diagram]</td>
<td></td>
</tr>
</tbody>
</table>

| 7. Built-in or fixed support     | A built-in or fixed support is capable of supporting an axial force $F$, a transverse force $V$ (shear force), and a couple $M$ (bending moment) to prevent rotation. |
| ![Built-in or fixed support]     | A                              |

| 8. Gravitational attraction      | The resultant of gravitational attraction on all elements of a body of mass $m$ is the weight $W = mg$ and acts toward the center of the earth through the center mass $G$. |
| ![Gravitational attraction]      | $W = mg$                      |

| 9. Spring action                | Spring force is tensile if spring is stretched and compressive if compressed. For a linearly elastic spring the stiffness $k$ is the force required to deform the spring a unit distance. |
| ![Spring action diagram]        |                                                        |

*Figure 3/1, continued* 

Body and is, therefore, distributed throughout it. The resultant of the gravitational forces on all elements is the weight $W = mg$ of the body, which passes through the center of mass $G$ and is directed toward the center of the earth for earthbound structures. The position of $G$ is frequently obvious from the geometry of the body, particularly where conditions of symmetry exist. When the position is not readily apparent, the location of $G$ must be calculated or determined by experiment. Similar remarks apply to the remote action of magnetic and electric forces. These forces of remote action have the same overall effect on a rigid body as forces of equal magnitude and direction applied by direct external contact.
Example 9 illustrates the action of a linear elastic spring and of a nonlinear spring with either hardening or softening characteristics. The force exerted by a linear spring, in tension or compression, is given by \( F = kx \), where \( k \) is the stiffness of the spring and \( x \) is its deformation measured from the neutral or undeformed position.

The student is urged to study these nine conditions and to identify them in the problem work so that the correct free-body diagrams may be drawn. The representations in Fig. 3/1 are not free-body diagrams but are merely elements in the construction of free-body diagrams.

The full procedure for drawing a free-body diagram which accomplishes the isolation of the body or system under consideration will now be described.

**Construction of free-body diagrams.** The following steps are involved.

**Step 1.** A clear decision is made concerning which body or combination of bodies is to be isolated. The body chosen will usually involve one or more of the desired unknown quantities.

**Step 2.** The body or combination chosen is next isolated by a diagram which represents its complete external boundary. This boundary defines the isolation of the body from all other contacting or attracting bodies, which are considered removed. This step is often the most crucial of all. We should always be certain that we have completely isolated the body before proceeding with the next step.

**Step 3.** All forces which act on the isolated body as applied by the removed contacting and attracting bodies are next represented in their proper positions on the diagram of the isolated body. A systematic traverse of the entire boundary will disclose all contact forces. Weights, where appreciable, must be included. Known forces should be represented by vector arrows, each with its proper magnitude, direction, and sense indicated. Each unknown force should be represented by a vector arrow with the unknown magnitude or direction indicated by symbol. If the sense of the vector is also unknown, it may be arbitrarily assumed. The calculations will reveal a positive quantity if the correct sense was assumed and a negative quantity if the incorrect sense was assumed. It is necessary to be consistent with the assigned characteristics of unknown forces throughout all of the calculations.

**Step 4.** The choice of coordinate axes should be indicated directly on the diagram. Pertinent dimensions may also be represented for convenience. Note, however, that the free-body diagram
serves the purpose of focusing accurate attention on the action of
the external forces, and therefore the diagram should not be clut-
tered with excessive extraneous information. Force arrows should
be clearly distinguished from any other arrows which may appear
so that confusion will not result. For this purpose a colored pencil
may be used.

When the foregoing four steps are completed, a correct free-
body diagram will result, and the way will be clear for a straight-
forward and successful application of the governing equations, both
in statics and in dynamics.

Many students are often tempted to omit from the free-body
diagram certain forces which may not appear at first glance to be
needed in the calculations. When we yield to this temptation, we
invite serious error. It is only through complete isolation and a sys-
tematic representation of all external forces that a reliable account-
ing of the effects of all applied and reactive forces can be made. Very
often a force which at first glance may not appear to influence a
desired result does indeed have an influence. Hence, the only safe
procedure is to make certain that all forces whose magnitudes are
not negligible appear on the free-body diagram.

The free-body diagram has been explained in some detail be-
cause of its great importance in mechanics. The free-body method
ensures an accurate definition of a mechanical system and focuses
attention on the exact meaning and application of the force laws of
statics and dynamics. Indeed, the free-body method is so important
that students are strongly urged to reread this section several times
in conjunction with their study of the sample free-body diagrams
shown in Fig. 3/2 and the sample problems which appear at the end
of the next article.

Figure 3/2 gives four examples of mechanisms and structures
together with their correct free-body diagrams. Dimensions and
magnitudes are omitted for clarity. In each case we treat the entire
system as a single body, so that the internal forces are not shown.
The characteristics of the various types of contact forces illustrated
in Fig. 3/1 are included in the four examples as they apply.

In example 1 the truss is composed of structural elements
which, taken all together, constitute a rigid framework. Thus, we
may remove the entire truss from its supporting foundation and
treat it as a single rigid body. In addition to the applied external
load \( P \), the free-body diagram must include the reactions on the
truss at \( A \) and \( B \). The rocker at \( B \) can support a vertical force only,
and this force is transmitted to the structure at \( B \) (example 4 of Fig.
3/1). The pin connection at \( A \) (example 6 of Fig. 3/1) is capable of
supplying both a horizontal and a vertical component of force to the
truss. In this relatively simple example it is clear that the vertical
component \( A_y \) must be directed down to prevent the truss from ro-
### SAMPLE FREE-BODY DIAGRAMS

<table>
<thead>
<tr>
<th>Mechanical System</th>
<th>Free-BODY Diagram of Isolated Body</th>
</tr>
</thead>
<tbody>
<tr>
<td><strong>1. Plane truss</strong></td>
<td><img src="image1" alt="Diagram 1" /></td>
</tr>
<tr>
<td>Weight of truss assumed negligible compared with $P$</td>
<td>$A_x$ $A_y$ $B_x$ $B_y$</td>
</tr>
<tr>
<td>$A$</td>
<td>$B$</td>
</tr>
<tr>
<td><strong>2. Cantilever beam</strong></td>
<td><img src="image2" alt="Diagram 2" /></td>
</tr>
<tr>
<td>$F_3$ $F_2$ $F_1$</td>
<td>$V$ $F_3$ $F_2$ $F_1$</td>
</tr>
<tr>
<td>$A$</td>
<td>$M$</td>
</tr>
<tr>
<td>Mass $m$</td>
<td>$W = mg$</td>
</tr>
<tr>
<td><strong>3. Beam</strong></td>
<td><img src="image3" alt="Diagram 3" /></td>
</tr>
<tr>
<td>Smooth surface contact at $A$. Mass $m$</td>
<td>$M$</td>
</tr>
<tr>
<td>$P$</td>
<td>$N$ $M$</td>
</tr>
<tr>
<td>$B$</td>
<td>$W = mg$</td>
</tr>
<tr>
<td><strong>4. Rigid system of interconnected bodies analyzed as a single unit</strong></td>
<td><img src="image4" alt="Diagram 4" /></td>
</tr>
<tr>
<td>$P$ Weight of mechanism neglected</td>
<td>$P$</td>
</tr>
<tr>
<td>$A$</td>
<td>$B$</td>
</tr>
<tr>
<td>$m$</td>
<td>$W = mg$</td>
</tr>
</tbody>
</table>

**Figure 3/2**

tating clockwise about $B$. Also, the horizontal component $A_x$ will be to the left to keep the truss from moving to the right under the influence of the horizontal component of $P$. Thus, in constructing the free-body diagram for this simple truss, the correct sense of each of the components of force exerted on the truss by the foundation at $A$ is easily perceived and may, therefore, be represented in its correct physical sense on the diagram. When the correct physical sense of a force or its component is not easily recognized by direct
observation, it must be assigned arbitrarily, and the correctness of
or error in the assignment is determined by the algebraic sign of its
calculated value. If the total weight of the truss members is appreci-able compared with \( P \) and the forces at \( A \) and \( B \), then the weights
of the members must be included on the free-body diagram as ex-
ternal forces.

In example 2 the cantilever beam is secured to the wall and
subjected to three applied loads. When we isolate that part of the
beam to the right of the section at \( A \), we must include the reactive
forces applied to the beam by the wall. The resultants of these re-
active forces are shown acting on the section of the beam (example
7 of Fig. 3/1). A vertical force \( V \) to counteract the excess of down-
ward applied force is shown, and a tension \( F \) to balance the excess
of applied force to the right must also be included. Then, to prevent
the beam from rotating about \( A \), a counterclockwise couple \( M \) is also
required. The weight \( mg \) of the beam must also be represented
through the mass center (example 8 of Fig. 3/1). Here we have rep-
resented the somewhat complex system of forces which actually act
on the cut section of the beam by the equivalent force–couple system
where the force is broken down into its vertical component \( V \) (shear
force) and its horizontal component \( F \) (tensile force). The couple \( M \)
is the bending moment in the beam. The free-body diagram is now
complete and shows the beam in equilibrium under the action of six
forces and one couple.

In example 3 the weight \( W = mg \) is shown acting through
the center of mass of the beam, which is assumed known (example 8
of Fig. 3/1). The force exerted by the corner \( A \) on the beam is normal
to the smooth surface of the beam (example 2 of Fig. 3/1). To per-
ceive this action more clearly, visualize an enlargement of the con-
tact point \( A \), which would appear somewhat rounded, and consider
the force exerted by this rounded corner on the straight surface of
the beam assumed to be smooth. If the contacting surfaces at the
corner were not smooth, a tangential frictional component of force
could be developed. In addition to the applied force \( P \) and couple \( M \),
there is the pin connection at \( B \), which exerts both an \( x \) and a \( y \)-component of force on the beam. The positive senses of these com-
ponents are assigned arbitrarily.

In example 4 the free-body diagram of the entire isolated mech-
anism discloses three unknown quantities for equilibrium with the
given loads \( mg \) and \( P \). Any one of many internal configurations for
securing the cable leading from the mass \( m \) would be possible with-
out affecting the external response of the mechanism as a whole,
and this fact is brought out by the free-body diagram. This hypo-
thetical example is used to emphasize the advantage of including as
much as possible in the free-body diagram and to show that the
forces internal to a rigid assembly of members do not influence the
values of the external reactions.
The positive senses of $B_x$ and $B_y$ in example 3 and $B_y$ in example 4 are assumed on the free-body diagrams, and the correctness of the assumptions would be proved or disproved according to whether the algebraic signs of the terms were plus or minus when the calculations were carried out in the actual problems.

The isolation of the mechanical system under consideration will be recognized as a crucial step in the formulation of the mathematical model. The most important aspect to the correct construction of the all-important free-body diagram is the clear-cut and unambiguous decision as to what is included and what is excluded. This decision becomes unambiguous only when the boundary of the free-body diagram represents a complete traverse of the body or system of bodies to be isolated, starting at some arbitrary point of the boundary and returning to that same point. The body within this closed boundary, then, is the isolated free body, and all forces transmitted to the body across the boundary by all contacting bodies which are removed must be accounted for. The student is again urged to devote special attention to this step. Before direct use is made of the free-body diagram in the application of the principles of force equilibrium in the next article, some initial practice with the drawing of free-body diagrams is helpful. The problems that follow are designed to provide this practice.
FREE-BODY DIAGRAM EXERCISES

3/A In each of the five following examples, the body to be isolated is shown in the left-hand diagram, and an incomplete free-body diagram (FBD) of the isolated body is shown on the right. Add whatever forces are necessary in each case to form a complete free-body diagram. The weights of the bodies are negligible unless otherwise indicated. Dimensions and numerical values are omitted for simplicity.

<table>
<thead>
<tr>
<th></th>
<th>Body</th>
<th>Incomplete FBD</th>
</tr>
</thead>
<tbody>
<tr>
<td>1.</td>
<td>Bell crank supporting mass $m$ with pin support at $A$. Flexible cable $A$</td>
<td><img src="image1" alt="Incomplete FBD 1" /></td>
</tr>
<tr>
<td>2.</td>
<td>Control lever applying torque to shaft at $O$. Pull $P$</td>
<td><img src="image2" alt="Incomplete FBD 2" /></td>
</tr>
<tr>
<td>3.</td>
<td>Boom OA, of negligible mass compared with mass $m$. Boom hinged at $O$ and supported by hoisting cable at $B$.</td>
<td><img src="image3" alt="Incomplete FBD 3" /></td>
</tr>
<tr>
<td>4.</td>
<td>Uniform crate of mass $m$ leaning against smooth vertical wall and supported on rough horizontal surface.</td>
<td><img src="image4" alt="Incomplete FBD 4" /></td>
</tr>
<tr>
<td>5.</td>
<td>Loaded bracket supported by pin connection at $A$ and fixed pin in smooth slot at $B$.</td>
<td><img src="image5" alt="Incomplete FBD 5" /></td>
</tr>
</tbody>
</table>

Figure 3/A
In each of the five following examples, the body to be isolated is shown in the left-hand diagram, and either a wrong or an incomplete free-body diagram (FBD) is shown on the right. Make whatever changes or additions are necessary in each case to form a correct and complete free-body diagram. The weights of the bodies are negligible unless otherwise indicated. Dimensions and numerical values are omitted for simplicity.

<table>
<thead>
<tr>
<th>Body</th>
<th>Wrong or Incomplete FBD</th>
</tr>
</thead>
<tbody>
<tr>
<td>1. Lawn roller of mass $m$ being pushed up incline $\theta$.</td>
<td><img src="image1" alt="Diagram of lawn roller" /></td>
</tr>
<tr>
<td>2. Pry bar lifting body A having smooth horizontal surface. Bar rests on horizontal rough surface.</td>
<td><img src="image2" alt="Diagram of pry bar" /></td>
</tr>
<tr>
<td>3. Uniform pole of mass $m$ being hoisted into position by winch. Horizontal supporting surface notched to prevent slipping of pole.</td>
<td><img src="image3" alt="Diagram of uniform pole" /></td>
</tr>
<tr>
<td>4. Supporting angle bracket for frame. Pin joints</td>
<td><img src="image4" alt="Diagram of angle bracket" /></td>
</tr>
<tr>
<td>5. Bent rod welded to support at $A$ and subjected to two forces and couple.</td>
<td><img src="image5" alt="Diagram of bent rod" /></td>
</tr>
</tbody>
</table>

Figure 3/B
Draw a complete and correct free-body diagram of each of the bodies designated in the statements. The weights of the bodies are significant only if the mass is stated. All forces, known and unknown, should be labeled. (Note: The sense of some reaction components cannot always be determined without numerical calculation.)

1. Uniform horizontal bar of mass \( m \) suspended by vertical cable at \( A \) and supported by rough inclined surface at \( B \).

2. Wheel of mass \( m \) on verge of being rolled over curb by pull \( P \).

3. Loaded truss supported by pin joint at \( A \) and by cable at \( B \).

4. Uniform bar of mass \( m \) and roller of mass \( m_0 \), taken together. Subjected to couple \( M \) and supported as shown. Roller is free to turn.

5. Uniform grooved wheel of mass \( m \) supported by a rough surface and by action of horizontal cable.

6. Bar, initially horizontal but deflected under load \( L \). Pinned to rigid support at each end.

7. Uniform heavy plate of mass \( m \) supported in vertical plane by cable \( C \) and hinge \( A \).

8. Entire frame, pulleys, and contact as cable to be isolated as a single unit.

Figure 3/C.